Mathematica Aeterna, Vol. 3, 2013, no. 2, 139 - 150

Solutions of Perturbed Nonlinear Nabla Fractional Difference Equations of Order $0 < \alpha < 1$

G.V.S.R.Deekshitulu

Department of Mathematics,
JNTU Kakinada,
Kakinada - 533003, Andhra Pradesh, India.
email: dixitgvsr@hotmail.com

J.Jagan Mohan

Fluid Dynamics Division,
School of Advanced Sciences,
VIT University,
Vellore - 632014, Tamil Nadu, India.
email: j.jaganmohan@hotmail.com

Abstract

The present paper provides methods and suitable criterion that describe the nature and behavior of solutions of nabla fractional difference equations of order $0 < \alpha < 1$, without actually constructing or approximating them. Since the existence and uniqueness of solutions of nabla discrete fractional order initial value problems is already guaranteed, we begin with the continuous dependence on the initial conditions and parameters. Next we develop a nonlinear variation of parameters formula and give an example.

Mathematics Subject Classification: 39A10, 39A99.

Keywords: Fractional order, difference equation, variation of parameters.

1 Introduction

Fractional calculus has gained importance during the past three decades due to its applicability in diverse fields of science and engineering and lot of literature is available on the applications of fractional calculus in modeling mechanical and electrical properties of real materials [13]. The notions of fractional calculus may be traced back to the works of Euler, but the idea of fractional difference is very recent.

Diaz and Osler [6] defined the fractional difference by the rather natural approach of allowing the index of differencing, in the standard expression for the n^{th} difference, to be any real or complex number. Later, Hirota [7], defined the fractional order difference operator ∇^{α} where α is any real number, using Taylor's series. Atsushi Nagai [11] adopted another definition for fractional difference by modifying Hirota's [7] definition. Recently, G.V.S.R.Deekshitulu and J.Jagan Mohan [4] modified the definition of Atsushi Nagai [11] and discussed some basic inequalities, comparison theorems and qualitative properties of the solutions of fractional difference equations [4, 5].

Throughout the present paper, we use the following notations: $\mathbb{N}$ is the set of natural numbers including zero and $\mathbb{Z}$ is the set of integers. $\mathbb{N}_a^+ = \{a, a+1, a+2, ...\}$ for $a \in \mathbb{Z}$. Let $u(n)$ be a real valued function defined on $\mathbb{N}_0^+$. Then for all $n_1, n_2 \in \mathbb{N}_0^+$ and $n_1 > n_2$, $\sum_{j=n_1}^{n_2} u(j) = 0$ and $\prod_{j=n_1}^{n_2} u(j) = 1$, i.e. empty sums and products are taken to be 0 and 1 respectively. If n and $n+1$ are in $\mathbb{N}_0^+$, the backward difference operator ∇ is defined as $\nabla u(n) = u(n) - u(n-1)$.

Now we introduce some basic definitions and results concerning nabla discrete fractional calculus. The extended binomial coefficient $\binom{a}{n}$, $(a \in \mathbb{R}, n \in \mathbb{Z})$ is defined by [11]

$$\binom{a}{n} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)} & \text{n} > 0 \\ 1 & \text{n} = 0 \\ 0 & \text{n} < 0. \end{cases}$$

In 2003, Atsushi Nagai [11] gave the following definition for fractional order difference operator.

Definition 1.1. *Let $\alpha \in \mathbb{R}$ and m be an integer such that $m-1 < \alpha \leq m$. The difference operator ∇ of order α, with step length ε, is defined as*

$$\nabla^{\alpha} u(n) = \begin{cases} \nabla^{\alpha-m}[\nabla^m u(n)] = \varepsilon^{m-\alpha} \sum_{j=0}^{n-1} \binom{\alpha-m}{j} (-1)^j \nabla^m u(n-j) & \alpha > 0 \\ u(n) & \alpha = 0 \\ \varepsilon^{-\alpha} \sum_{j=0}^{n-1} \binom{\alpha}{j} (-1)^j u(n-j) & \alpha < 0. \end{cases}$$

The above definition of $\nabla^{\alpha} u(n)$ given by Atsushi Nagai [11] contains ∇ operator and the term $(-1)^j$ inside the summation index and hence it becomes difficult to study the properties of solution. G.V.S.R.Deekshitulu and J.Jagan Mohan [4] modified the above definition for $\alpha \in (0, 1)$ as follows.

Definition 1.2. *The fractional sum operator of order α ($\alpha \in \mathbb{R}$) is defined as*

$$\nabla^{-\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j+\alpha-1}{j} u(n-j) = \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} u(j) \qquad (1)$$

and the fractional difference operator of order α ($\alpha \in \mathbb{R}$ and $0 < \alpha < 1$) is defined as

$$\nabla^{\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla u(n-j) = \sum_{j=1}^{n} \binom{n-j-\alpha-1}{n-j} u(j) - \binom{n-\alpha-1}{n-1} u(0). \tag{2}$$

Remark 1.3. *If we take $\alpha = 1$ in (2), using the definition of generalized binomial co-efficient, we have*

$$\begin{aligned}
\sum_{j=0}^{n-1} \binom{j-1}{j} \nabla u(n-j) &= \binom{-1}{0} \nabla u(n) + \sum_{j=1}^{n-1} \binom{j-1}{j} \nabla u(n-j) \\
&= \binom{-1}{0} \nabla u(n) + \sum_{j=1}^{n-1} \binom{j-1}{-1} \nabla u(n-j) \\
&= \nabla u(n).
\end{aligned}$$

Remark 1.4. *Let $u(n)$ and $v(n)$ be real valued functions defined on $\mathbb{N}_0^+$ and α, $\beta \in \mathbb{R}$ such that $0 < \alpha, \beta, \alpha + \beta < 1$ and c, d are constants. Then*

1. $\nabla^{\beta} \nabla^{\gamma} u(n) = \nabla^{\beta+\gamma} u(n)$.
2. $\nabla^{\alpha}[cu(n) + dv(n)] = c\nabla^{\alpha} u(n) + d\nabla^{\alpha} v(n)$.
3. $\nabla^{-\alpha} \nabla^{\alpha} u(n) = u(n) - u(0)$.
4. $\nabla^{\alpha} \nabla^{-\alpha} u(n) = u(n)$.
5. $\nabla^{\alpha} u(0) = 0$ *and* $\nabla^{\alpha} u(1) = u(1) - u(0) = \nabla u(1)$.

Remark 1.5. *[3] If p is any positive integer then, for $0 < \alpha < 1$,*

$$\nabla^{\alpha} n^{(p)} = \Gamma(p+1) \binom{n-\alpha}{n-p}.$$

Definition 1.6. *Let $u(n) : \mathbb{N}_0^+ \to \mathbb{R}$ and $f(n, r) : \mathbb{N}_0^+ \times \mathbb{R} \to \mathbb{R}$. Then a nonlinear difference equation of order α, $0 < \alpha < 1$, together with an initial condition is of the form*

$$\nabla^{\alpha} u(n+1) = f(n, u(n)), \; u(0) = u_0. \tag{3}$$

The existence and uniqueness of solutions to difference equations is obvious as the solutions are expressed as recurrence relations involving the values of the unknown function at the previous arguments as follows:

$$u(n) = u_0 + \sum_{j=0}^{n-1} B(n-1, \alpha; j) f(j, u(j)) \tag{4}$$

where $B(n,\alpha;j) = \binom{n-j+\alpha-1}{n-j}$ for $0 \le j \le n$. Recently, the authors have established the following fractional order discrete Gronwall-Bellman type inequality [5].

Theorem 1.7. *Let $u(n)$, $a(n)$ and $b(n)$ be real valued nonnegative functions defined on $\mathbb{N}_0^+$. If, for $0 < \alpha < 1$,*

$$u(n) \le u_0 + \sum_{j=0}^{n-1} B(n-1,\alpha;j)[a(j)u(j) + b(j)] \tag{5}$$

then

$$u(n) \le u_0 \prod_{j=0}^{n-1} \Big[1+B(n-1,\alpha;j)a(j)\Big] + \sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j) \prod_{k=j+1}^{n-1} \Big[1+B(n-1,\alpha;j)a(k)\Big] \tag{6}$$

for $n \in \mathbb{N}_0^+$.

In this present work, the differentiability properties with respect to the initial conditions of the solution of the difference equation of order α $(0 < \alpha < 1)$ are discussed. Then the solution of the perturbed equation is obtained.

2 Dependence on Initial Conditions and Parameters

The initial value problem (3) describes a model of a physical problem in which often some parameters such as lengths, masses, temperature etc. are involved. The values of these parameters can be measured only up to a certain degree of accuracy. Thus, in (3) the initial value u_0 as well as the function $f(n,u(n))$ may be subject to some errors either by necessity or for convenience. Hence, it is important to know how the solution changes when u_0 and $f(n,u(n))$ arc slightly altered. We shall discuss this question quantitatively in the following:

Theorem 2.1. *Let the following conditions be satisfied.*

1. *$f(n,u(n))$ is defined on $\mathbb{N}_0^+ \times \mathbb{R}$ and for all $(n,u(n)),(n,v(n)) \in \mathbb{N}_0^+ \times \mathbb{R}$*

$$|f(n,u(n)) - f(n,v(n))| \le \lambda(n)|u(n) - v(n)| \tag{7}$$

where $\lambda(n)$ is a nonnegative function defined on $\mathbb{N}_0^+$.

2. *$g(n,u(n))$ is defined on $\mathbb{N}_0^+ \times \mathbb{R}$ and for all $(n,u(n)) \in \mathbb{N}_0^+ \times \mathbb{R}$*

$$|g(n,u(n))| \le \mu(n) \tag{8}$$

where $\mu(n)$ is a nonnegative function defined on $\mathbb{N}_0^+$.

Then, for the solutions $u(n)$ and $v(n)$ of the initial value problems (3) and

$$\nabla^{\alpha} v(n+1) = f(n, v(n)) + g(n, v(n)), \ v(0) = v_0, \tag{9}$$

the following inequality holds

$$\begin{aligned} |u(n) - v(n)| \ &\leq \ |u_0 - v_0| \prod_{j=0}^{n-1} \Big[1 + B(n-1, \alpha; j)\lambda(j)\Big] \\ &+ \ \sum_{j=0}^{n-1} B(n-1, \alpha; j)\mu(j) \prod_{k=j+1}^{n-1} \Big[1 + B(n-1, \alpha; j)\lambda(k)\Big] \end{aligned} \tag{10}$$

Proof. Using (4), the initial value problems (3) and (9) are equivalent to

$$u(n) = u_0 + \sum_{j=0}^{n-1} B(n-1, \alpha; j) f(j, u(j)) \tag{11}$$

and

$$v(n) = v_0 + \sum_{j=0}^{n-1} B(n-1, \alpha; j)[f(j, v(j)) + g(j, v(j))]. \tag{12}$$

Then

$$u(n) - v(n) = u_0 - v_0 + \sum_{j=0}^{n-1} B(n-1, \alpha; j)[f(j, u(j)) - f(j, v(j))] - \sum_{j=0}^{n-1} B(n-1, \alpha; j) g(j, v(j)). \tag{13}$$

Thus, from (7) and (8) it, follows that

$$|u(n) - v(n)| \leq |u_0 - v_0| + \sum_{j=0}^{n-1} B(n-1, \alpha; j)[\lambda(j)|u(j) - v(j)| + \mu(j)]. \tag{14}$$

Now an application of Theorem 1.7 yields (10). □

Hereafter, to emphasize the dependence of the initial point $(0, u_0)$ we shall denote the solutions of the initial value problems (1.5) as $u(n, 0, u_0)$. In our next result we shall show that $u(n, 0, u_0)$ is differentiable with respect to u_0.

Theorem 2.2. *Let for all $(n, u(n)) \in \mathbb{N}_0^+ \times \mathbb{R}$ the function $f(n, u(n))$ be defined and the partial derivative $\frac{\partial f}{\partial u}$ exist. Further, let the solution $u(n) = u(n, 0, u_0)$ of the initial value problem (3) exist on $\mathbb{N}_0^+$ and*

$$H(n, 0, u_0) = \frac{\partial f(n, u(n, 0, u_0))}{\partial u} \tag{15}$$

then,

$$\Phi(n,0,u_0)=\frac{\partial u(n,0,u_0)}{\partial u_0} \tag{16}$$

exists and is the solution of the initial value problem

$$\nabla^{\alpha}\Phi(n+1,0,u_0)=H(n,0,u_0)\Phi(n,0,u_0),\ \ \Phi(n_0,0,u_0)=I. \tag{17}$$

Proof. Since $u(n,0,u_0)$ is the solution of (3), we have

$$\nabla^{\alpha}u(n+1,0,u_0)=f(n,u(n,0,u_0)),\ \ u(n,0,u_0)=u_0. \tag{18}$$

Consider

$$\begin{aligned}\frac{\partial}{\partial u_0}[\nabla^{\alpha}u(n+1,0,u_0)] &= \lim_{h\to 0}\frac{1}{h}\Big[\nabla^{\alpha}u(n+1,0,u_0+h)-\nabla^{\alpha}u(n+1,0,u_0)\Big]\\ &= \nabla^{\alpha}\Big[\lim_{h\to 0}\frac{u(n+1,0,u_0+h)-u(n+1,0,u_0)}{h}\Big]\\ &= \nabla^{\alpha}\Big[\frac{\partial}{\partial u_0}u(n+1,0,u_0)\Big]\\ &= \nabla^{\alpha}\Phi(n+1,0,u_0)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial u_0}[f(n,u(n,0,u_0))] &= \frac{\partial}{\partial u}[f(n,u(n,0,u_0))]\frac{\partial}{\partial u_0}[u(n,0,u_0)]\\ &= H(n,0,u_0)\Phi(n,0,u_0). \end{aligned} \tag{19}$$

Thus, in view of (19), we get (17). □

Theorem 2.3. *Assume that*

$$|f(n,u(n))-f(n,v(n))|\le g(n,|u(n)-v(n)|) \text{ for all } (n,u(n)),(n,v(n))\in\mathbb{N}_0^+\times\mathbb{R}, \tag{20}$$

where $g(n,r)$ is defined on $\mathbb{N}_0^+\times\mathbb{R}^+$ and nondecreasing in r for any fixed $n\in\mathbb{N}_0^+$. Further, let $u(n,0,u_1)$ and $u(n,0,u_2)$ be solutions of (3) exist on $\mathbb{N}_0^+$. Then, for all $n\in\mathbb{N}_0^+$,

$$|u(n,0,u_1)-u(n,0,u_2)|\le r(n,0,r_0), \tag{21}$$

where $r(n)=r(n,0,r_0)$ is the solution of the initial value problem

$$\nabla^{\alpha}r(n+1)=g(n,r(n)),\ \ r(0)=r_0(\ge|u_1-u_2|). \tag{22}$$

Proof. Since $u(n,0,u_1)$ and $u(n,0,u_2)$ are solutions of (3), we get

$$u(n,0,u_1)=u_1+\sum_{j=0}^{n-1}B(n-1,\alpha;j)f(j,u(j,0,u_1)) \tag{23}$$

and

$$u(n,0,u_2) = u_2 + \sum_{j=0}^{n-1} B(n-1,\alpha;j) f(j,u(j,0,u_2)). \tag{24}$$

Then

$$|u(n,0,u_1)-u(n,0,u_2)| \le |u_1-u_2| + \sum_{j=0}^{n-1} B(n-1,\alpha;j)|f(j,u(j,0,u_1))-f(j,u(j,0,u_2))|. \tag{25}$$

Let $z(n) = |u(n,0,u_1) - u(n,0,u_2)|$. Then, using (20),

$$z(n) \le z(0) + \sum_{j=0}^{n-1} B(n-1,\alpha;j) g(j,z(j)).$$

Further, since $z(0) \le r(0)$ and

$$r(n) = r(0) + \sum_{j=0}^{n-1} B(n-1,\alpha;j) g(j,r(j)),$$

the inequality (21) follows by induction. □

Remark 2.4. *If $r(n,0,0) = 0$ for all $n \in \mathbb{N}_0^+$ and $r(n,0,r_0) \to 0$ as $r_0 \to 0$, then from (21) it is clear that the solution $r(n,0,u_0)$ continuously depends on u_0.*

Now we shall consider the following initial value problem.

$$\nabla^\alpha u(n+1) = f(n,u(n),p(n)), \;\; u(0) = u_0 \tag{26}$$

where $p(n) \in \mathbb{R}$ is a parameter such that $|p(n) - p_0| \le \delta(> 0)$ and p_0 is a fixed scalar in $\mathbb{R}$. For a given $p(n)$ such that $|p(n) - p_0| \le \delta$ we shall assume that the solution $u(n,p(n)) = u(n,0,u_0,p(n))$ of (26) exists on $\mathbb{N}_0^+$.

Theorem 2.5. *Let for all $n \in \mathbb{N}_0^+$, $u(n)$, $p(n) \in \mathbb{R}$ such that $|p(n) - p_0| \le \delta$ the function $f(n,u(n),p(n))$ is defined, and the following inequalities hold*

$$|f(n,u(n),p(n)) - f(n,v(n),p(n))| \le \lambda(n)|u(n) - v(n)| \tag{27}$$

and

$$|f(n,u(n),p_1) - f(n,u(n),p_2)| \le \mu(n)|p_1 - p_2|, \tag{28}$$

where $\lambda(n)$ *and* $\mu(n)$ *are nonnegative functions defined on* $\mathbb{N}_0^+$. *Then, for the solutions* $u(n,0,u_1,p_1)$ *and* $u(n,0,u_2,p_2)$ *of (26) the following inequality holds*

$$\begin{aligned}|u(n,0,u_1,p_1)-u(n,0,u_2,p_2)| \;\leq\; & |u_1-u_2|\prod_{j=0}^{n-1}\Big[1+B(n-1,\alpha;j)\lambda(j)\Big] \\ & +|p_1-p_2|\sum_{j=0}^{n-1}\Big[B(n-1,\alpha;j)\mu(j) \\ & \prod_{k=j+1}^{n-1}\Big[1+B(n-1,\alpha;j)\lambda(k)\Big]\Big]. \qquad (29)\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 2.1. □

Theorem 2.6. *Let for all* $n\in\mathbb{N}_0^+$, $u(n)$, $p(n)\in\mathbb{R}$ *such that* $|p(n)-p_0|\leq\delta$ *the function* $f(n,u(n),p(n))$ *is defined, and the partial derivatives* $\frac{\partial f}{\partial u}$ *and* $\frac{\partial f}{\partial p}$ *exist. Further, if* $u(n,p(n))=u(n,0,u_0,p(n))$ *is the solution of of (26) on* $\mathbb{N}_0^+$ *then,*

$$\Phi(n,0,u_0,p(n))=\frac{\partial u(n,0,u_0,p(n))}{\partial p} \qquad (30)$$

exists and is the solution of the initial value problem

$$\begin{aligned}\nabla^{\alpha}\Phi(n,0,u_0,p(n)) &= G(n,0,u_0,p(n))\Phi(n,0,u_0,p(n))+H(n,0,u_0,p(n)),\\ \Phi(0,0,u_0,p(n)) &= 0 \qquad (31)\end{aligned}$$

where

$$G(n,0,u_0,p(n))=\frac{\partial f(n,u(n,p),p)}{\partial u} \qquad (32)$$

and

$$H(n,0,u_0,p(n))=\frac{\partial f(n,u(n,p),p)}{\partial p}. \qquad (33)$$

Proof. The proof is similar to the proof of Theorem 2.2. □

3 Method of Nonlinear Variation of Parameters

The main purpose of this section is to develop the variation of parameters formula to represent the solution $v(n,0,u_0)$ of the perturbed problem (9) in terms of the solution $u(n,0,u_0)$ of the unperturbed problem (3).

Theorem 3.1. *Let for all $n \in \mathbb{N}_0^+$, $u(n) \in \mathbb{R}$ the functions $f(n,u(n))$ and $g(n,u(n))$ be defined, $\frac{\partial f}{\partial u}$ exist and continuous and invertible. If for each $u_0 \in \mathbb{R}$ the solution $u(n,0,u_0)$ of (3) exists on $\mathbb{N}_0^+$ and $\Phi(n,0,u_0) = \frac{\partial u(n,0,u_0)}{\partial u_0}$ is as defined in Theorem 2.2, then any solution $v(n) = v(n,0,u_0)$ of (9) satisfies the equation*

$$v(n,0,u_0) = u\Big(n,0,u_0+\sum_{i=0}^{n-1}\Big[\Psi^{-1}(i+1,0,w(i),w(i+1))\Big(\sum_{j=0}^{i}B(i,\alpha;j)g(j,v(j))\Big)\Big]\Big) \tag{34}$$

where

$$\Psi(i,0,w(k),w(k+1)) = \int_0^1 [\Phi(i,0,sw(k+1)+(1-s)w(k))]ds, \tag{35}$$

$w(n)$ satisfies the implicit equation

$$w(n) = u_0 + \sum_{i=0}^{n-1}\Psi^{-1}(i+1,0,w(i),w(i+1))\Big[\sum_{j=0}^{i}B(i,\alpha;j)g(j,v(j))\Big]. \tag{36}$$

Proof. The solution of (9) is given by

$$v(n+1,0,u_0) = v(0,0,u_0) + \sum_{j=0}^{n}B(n,\alpha;j)[f(j,v(j,0,u_0)) + g(j,v(j,0,u_0))]. \tag{37}$$

The method of variation of parameters requires determining a function $w(n)$ so that $v(n,0,u_0) = u(n,0,w(n))$, $w(0) = u_0$. Then, from (37), we have

$$u(n+1,0,w(n+1)) = u(0,0,u_0)+\sum_{j=0}^{n}B(n,\alpha;j)[f(j,u(j,0,w(j)))+g(j,u(j,0,w(j)))]. \tag{38}$$

Since $u(n,0,u_0)$ is the solution of (3), we have

$$u(n+1,0,w(n)) = u(0,0,u_0) + \sum_{j=0}^{n}B(n,\alpha;j)f(j,u(j,0,w(j))). \tag{39}$$

Using (38) and (39), we get

$$\sum_{j=0}^{n}B(n,\alpha;j)g(j,u(j,0,w(j))) = u(n+1,0,w(n+1)) - u(n+1,0,w(n)). \tag{40}$$

Using mean value theorem, we get

$$\begin{aligned}\int_0^1 \frac{\partial}{\partial u_0}u(n+1,0,sw(n+1)+(1-s)w(n))ds &= \frac{u(n+1,0,w(n+1)) - u(n+1,0,w(n))}{w(n+1)-w(n)}\\ \text{or } \Psi(n+1,0,w(n),w(n+1))\nabla w(n+1) &= \sum_{j=0}^{n}B(n,\alpha;j)g(j,v(j)).\end{aligned}$$

Thus

$$w(n) = u_0 + \sum_{i=0}^{n-1} \Big[\Psi^{-1}(i+1, 0, w(i), w(i+1))\Big(\sum_{j=0}^{i} B(i, \alpha; j)g(j, v(j))\Big)\Big].$$

Hence the proof. □

Corollary 3.2. *Let the assumptions of Theorem 3.1 be satisfied. Then,*

$$\begin{aligned} v(n, 0, u_0) &= u(n, 0, u_0) + \sum_{i=0}^{n-1} \Big(\Psi(n, 0, w(n), u_0)\Psi^{-1}(i+1, 0, w(i), w(i+1)) \\ &\quad \Big[\sum_{j=0}^{i} B(i, \alpha; j)g(j, v(j))\Big]\Big). \end{aligned} \tag{41}$$

Proof. Using mean value theorem, we get

$$\begin{aligned} \frac{u(n, 0, w(n)) - u(n, 0, u_0)}{w(n) - u_0} &= \int_0^1 \frac{\partial}{\partial u_0}[u(n, 0, sw(n) + (1-s)u_0)]ds \\ \text{or } u(n, 0, w(n)) &= u(n, 0, u_0) + [w(n) - u_0]\Psi(n, 0, w(n), u_0). \end{aligned} \tag{42}$$

Using (34) and (36) in (42), we get (41). □

Example 3.3. *Solve* $\nabla^\alpha v(n+1) = v(n) + n$, $v(0) = v_0$.

Solution: Let $u(n, 0, u_0)$ be the solution of unperturbed problem $\nabla^\alpha u(n+1) = u(n)$, $u(0) = u_0$. Then, from (4),

$$u(n, 0, u_0) = u_0 \prod_{j=0}^{n-1} \Big[1 + B(n-1, \alpha; j)\Big]. \tag{43}$$

Let $v(n, 0, u_0)$ be the solution of perturbed problem $\nabla^\alpha v(n+1) = v(n) + n$, $v(0) = u(0) = u_0$. Take $v(n, 0, u_0) = u(n, 0, w(n))$. Then, from (43),

$$v(n, 0, u_0) = w(n) \prod_{j=0}^{n-1} \Big[1 + B(n-1, \alpha; j)\Big]. \tag{44}$$

Now $\Phi(n, 0, u_0) = \frac{\partial}{\partial u_0} u(n, 0, u_0) = \prod_{j=0}^{n-1}[1 + B(n-1, \alpha; j)]$. Then $\Phi(n + 1, 0, sw(k+1) + (1-s)w(k)) = \prod_{j=0}^{n} \Big[1 + B(n, \alpha; j)\Big]$.
Further $\Psi(n+1, 0, w(n), w(n+1)) = \prod_{j=0}^{n} \Big[1 + B(n, \alpha; j)\Big]$ and $\Psi^{-1}(n +$

$1, 0, w(n), w(n+1)) = \prod_{j=0}^{n} \Big[1 + B(n, \alpha; j)\Big]^{-1}$. Thus, using (36), we have

$$\begin{aligned} w(n) &= u_0 + \sum_{i=0}^{n-1} \Big(\Psi^{-1}(i+1, 0, w(i), w(i+1))\Big[\sum_{j=0}^{i} B(i, \alpha; j) g(j, v(j))\Big]\Big) \\ &= u_0 + \sum_{i=0}^{n-1} \Big[\Big(\prod_{j=0}^{i} \Big[1 + B(i, \alpha; j)\Big]^{-1}\Big)\Big(\sum_{j=0}^{i} jB(i, \alpha; j)\Big)\Big]. \end{aligned}$$

But, using Remark 1.5,

$$\sum_{j=0}^{i} jB(i, \alpha; j) = \sum_{j=0}^{i} B(i, \alpha; j) j^{(1)} = \Gamma(2)\binom{i+\alpha}{i-1}.$$

Therefore

$$w(n) = u_0 + \Gamma(2) \sum_{i=0}^{n-1} \Big[\prod_{j=0}^{i} \Big(1 + B(i, \alpha; j)\Big)^{-1} \binom{i+\alpha}{i-1}\Big].$$

Hence from (44),

$$v(n) = u(n, 0, w(n)) = \Big[u_0 + \Gamma(2) \sum_{i=0}^{n-1} \Big[\prod_{j=0}^{i} \Big[1 + B(i, \alpha; j)\Big]^{-1} \binom{i+\alpha}{i-1}\Big]\Big] \prod_{j=0}^{n-1} \Big[1 + B(n-1, \alpha; j)\Big]$$

is the solution of given nonlinear fractional order difference equation.

ACKNOWLEDGEMENTS. The authors are grateful to the referees for their suggestions and comments which considerably helped improve the content of this paper.

References

[1] Agarwal, R.P. *Difference equations and inequalities*, Marcel Dekker, New York, 1992.

[2] Alekseev, V.M. *An estimate for the perturbations of the solutions of ordinary differential equations*, Vestnik Moskov. Univ. Ser. I Mat. Mekh., 2 (1961), 28 - 36.

[3] Deekshitulu, G.V.S.R. and Jagan Mohan, J. *Discrete fractional calculus*, Rev. Bull. Cal. Math. Soc, 19 (2011), No. 2, 173 - 184.

[4] Deekshitulu, G.V.S.R. and Jagan Mohan, J. *Fractional difference inequalities*, Communications in Applied Analysis, 14 (2010), No. 1, 89 - 98.

[5] Deekshitulu, G.V.S.R. and Jagan Mohan, J. *Some new fractional difference inequalities*, ICMMSC 2012, CCIS 283 (2012), Springer-Verlag, Berlin, Heidelberg, 403 - 412.

[6] Diaz, J.B. and Osler, T.J. *Differences of fractional order*, Math. Comp., 28 (1974), 185 - 201.

[7] Hirota, R. *Lectures on difference equations*, Science-sha, 2000 (in Japanese).

[8] Lakshmikantham, V. and Deo, S.G. *Method of Variation of Parameters for Dynamic Systems*, Gordon and Breach, Amsterdam, 1998.

[9] Lakshmikantham, V. and Leela, S. *Differential and integral inequalities*, Vol. I, Academic Press, New York, 1969.

[10] *Lakshmikantham, V. and Trigiante, D. Theory of difference equations*, Academic Press, New York, 1988.

[11] Nagai, A. *An integrable mapping with fractional difference*, J. Phys. Soc. Jpn., 72 (2003), 2181 - 2183.

[12] Pachpatte, B.G. *Inequalities of finite difference equations*, Marcel Dekker, New York, 2002.

[13] Podlubny, I *Fractional differential equations*, Academic Press, San Diego, 1999.

Received: February, 2013