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ON SOME NEW I-CONVERGENT SEQUENCE SPACES

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Abstract

In this article we introduce the sequence spaces $V_{0\sigma}^{I}(m,\epsilon)$ and $V_{\sigma}^{I}(m,\epsilon)$ and study some of the properties and inclusion relations on these spaces.

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1 Introduction

Let , and be the sets of all natural,real and complex numbers respectively. We write

$$\omega = \{x = (x_k) : x_k \in \ or \ \},$$

the space of all real or complex sequences.

Let l_∞, c and c_0 denote the Banach spaces of bounded,convergent and nul sequences respectively normed by

$$||x||_\infty = \sup_k |x_k|$$

.

Let v denote the space of sequences of bounded variation ,that is

$$v = \{x = (x_k) : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty, x_{-1} = 0\}.$$

v is a Banach space normed by

$$||x|| = \sum_{k=0}^{\infty} |x_k - x_{k-1}| (See[5], [7], [12], [14]).$$

Let σ be an injection of the set of positive integers into itself having no finite orbits and T be the operator defined on l_∞ by $T(x_k) = (x_{\sigma(k)})$.

A positive linear functional functional Φ, with $||\Phi|| = 1$, is called a σ-mean or an invariant mean if $\Phi(x) = \Phi(Tx)$ for all $x \in l_\infty$.

A sequence x is said to be σ-convergent , denoted by $x \in V_\sigma$, if $\Phi(x)$ takes the same value , called $\sigma - \lim x$, for all σ-means Φ. We have

$$V_\sigma = \{x = (x_k) : \sum_{m=1}^{\infty} t_{m,k}(x) = L \ \text{ uniformly in } k, \text{ L} = \sigma - \lim x\},$$

where for $m \geq 0, k > 0$

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} + ... + x_{\sigma^m(k)}}{m+1}, \text{and } t_{-1,k} = 0$$

where $\sigma^m(k)$ denotes the mth iterate of σ at k.

In particular , if σ is the translation, a σ-mean is often called a Banach limit and V_σ reduces to f, the set of almost convergent sequences.(See[6],[7],[8],[14]).For certain kinds of mappings σ,every invariant mean Φ extends the limit functional on the space c of real convergent sequences, in the sense that

$$\Phi(x) = \lim x \ \text{ for all } \ x \in c.$$

Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all of whose σ-mean are equal.(cf.[1],[5],[6],[7],[8],[11],[12],[14],[15],[16]).

The notion of I-convergence was studied at the initial stage by Kostyrko[4],Šalát[4] and Wilczyński[4]. Later on it was studied by Šalát[9-10], Tripathy[9-10], Ziman[9-10],Tripathy and Hazarika[13] and Demirci[2].Here we give some preliminaries about the notion of ideal convergence.

Let X be a non empty set. Then a family of sets I$\subseteq 2^X$(power set of X)is said to be an ideal if I is additive i.e A,B$\in$I $\Rightarrow$A$\cup$ B$\in$I and hereditary i.e A$\in$I, B⊆A⇒B∈I.

A non-empty family of sets $\pounds \subseteq 2^X$ is said to be filter on X if and only if $\phi \notin \pounds$(I),for A,B∈ $\pounds$(I) we have A∩B∈ $\pounds$(I) and for each A∈ $\pounds$(I)and A⊆B implies B∈ $\pounds$(I).

An ideal I⊆ 2^X is called non-trivial if I≠ 2^X.

A non-trivial ideal I⊆ 2^X is called admissible if $\{\{x\} : x \in X\}$ ⊆I.

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal J≠I containing I as a subset.

For each ideal I, there is a filter $\pounds$(I) corresponding to I.
i.e $\pounds$(I) $= \{K \subseteq: K^c \in I\}$,where $K^c = -K$ (See.[13]).

Definition.1.1 A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$. $\{k \in N : |x_k - L| \geq \epsilon\} \in$ I. In this case we write $I - \lim x_k = L$.

The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in: |x_k - L| \geq \epsilon\} \in I, \text{for some L}\in\}\text{(See.[4],[9],[10])}.$$

Definition.1.2 A sequence $(x_k) \in \omega$ is said to be I-null if $L = 0$ .In this case we write $I - \lim x_k = 0$.The space c_0^I of I-null sequences is given by

$$c_0^I = \{(x_k) \in \omega : \{k \in: |x_k| \geq \epsilon\} \in I, \}\text{(See.[4],[9],[10])}.$$

Definition.1.3 A sequence $(x_k) \in \omega$ is said to be I-Cauchy if for every $\epsilon > 0$ there exists a number m = m(ϵ) such that $\{k \in: |x_k - x_m| \geq \epsilon\} \in$ I.(See.[13]).

Definition.1.4 A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists M >0 such that $\{k \in: |x_k| > M\}$(See.[13]).

Definition.1.5 A sequence space E is said to be solid or normal if $(x_k) \in$ E implies $(\alpha_k x_k) \in$ E for all sequence of scalars (α_k) with $|\alpha_k| < 1$ for all $k \in$.(See.[13]).

Definition.1.6 A sequence space E is said to be monotone if it contains the cannonical preimages of all its stepspaces.(See[13]).

The following result will be used for establishing some results of this article

Lemma.1.7 The sequence space E is solid implies that E is monotone.(See[3,p.53]).

The motivation for this paper comes from the study of [1-16] and here we generalise the notion of the $\sigma-$mean using I-convegence.

2 Main Results

In this article we introduce the following classes of sequence spaces.

Let $x = (x_k) \in \omega$,

$$V^I_{0\sigma}(m,\epsilon) = \{(x_k) \in \omega : (\forall m)(\exists \epsilon > 0)\{k \in : |t_{m,k}(x)| \geq \epsilon\} \in I\},$$

$$V^I_{\sigma}(m,\epsilon) = \{(x_k) \in \omega : (\forall m)(\exists \epsilon > 0)\{k \in : |t_{m,k}(x) - L| \geq \epsilon\} \in I, \text{for some L} \in\}.$$

Theorem 2.1.$V^I_{\sigma}(m,\epsilon)$ and $V^I_{0\sigma}(m,\epsilon)$ are linear spaces.

Proof: Let $(x_k),(y_k) \in V^I_{\sigma}(m,\epsilon)$ and α,β be two scalars. Then for a given $\epsilon > 0$,
we have

$$A_1 = \{k \in : |t_{m,k}(x) - L_1| < \frac{\epsilon}{2}\} \in I, \text{for some L}_1 \in\}$$

$$A_2 = \{k \in : |t_{m,k}(y) - L_2| < \frac{\epsilon}{2}\} \in I, \text{for some L}_2 \in\}$$

Then

$$A_1^c = \{k \in : |t_{m,k}(x) - L_1| \geq \frac{\epsilon}{2}\} \in I, \text{for some L}_1 \in\}$$

$$A_2^c = \{k \in : |t_{m,k}(y) - L_2| \geq \frac{\epsilon}{2}\} \in I, \text{for some L}_2 \in\}$$

Now let,

$$A_3 = \{k \in : |(\alpha t_{m,k}(x) + \beta t_{m,k}(y)) - (\alpha L_1 + \beta L_2)| < \epsilon\}$$

$$\supseteq \{k \in : |\alpha||t_{m,k}(x) - L_1| < \epsilon\} \cap \{k \in : |\beta||t_{m,k}(y) - L_2| < \epsilon\}$$

Thus $A_3^c = A_1^c \cap A_2^c \in I$.
Hence$(\alpha(x_k) + \beta(y_k)) \in V^I_{\sigma}(m,\epsilon)$.
Therefore $V^I_{\sigma}(m,\epsilon)$ is a linear space.
The rest of the result follow similarly.

Theorem 2.2. The spaces $V_{0\sigma}^{I}(m,\epsilon)$ and $V_{\sigma}^{I}(m,\epsilon)$ are normed linear spaces,normed by

$$||x_k||_* = \sup_{m,k} |t_{m,k}(x)|. \qquad (A).$$

Proof: It is clear from from theorem 2.1 that $V_{0\sigma}^{I}(m,\epsilon)$ and $V_{\sigma}^{I}(m,\epsilon)$ are linear spaces.

It is easy to verify that (A) defines a norm on the spaces $V_{0\sigma}^{I}(m,\epsilon)$ and $V_{\sigma}^{I}(m,\epsilon)$.

Theorem 2.3. $V_{\sigma}^{I}(m,\epsilon)$ is a closed subspace of l_∞.

Proof. Let $(x_k^{(n)})$ be a cauchy sequence in $V_{\sigma}^{I}(m,\epsilon)$ such that $x^{(n)} \to x$.
We show that $x \in V_{\sigma}^{I}(m,\epsilon)$. Since $(x_k^{(n)}) \in V_{\sigma}^{I}(m,\epsilon)$, then there exists a_n such that

$$\{k \in : |t_{m,k}(x^{(n)}) - a_n| \geq \epsilon\} \in I.$$

We need to show that
(1)(a_n) converges to a.
(2)If $U = \{k \in : |x_k - a| < \epsilon\}$, then $U^c \in I$.

(1) Since $(x_k^{(n)})$ is a cauchy sequence in $V_{\sigma}^{I}(m,\epsilon)$ then for a given $\epsilon > 0$, there exists $k_0 \in$ such that

$$\sup_{m,k} |t_{m,k}(x_k^{(n)}) - t_{m,k}(x_k^{(i)})| < \frac{\epsilon}{3}, \text{for all n,i} \geq k_0$$

For a given $\epsilon > 0$, we have

$$B_{ni} = \{k \in : |t_{m,k}(x_k^{(n)}) - t_{m,k}(x_k^{(i)})| < \frac{\epsilon}{3}\}$$

$$B_i = \{k \in : |t_{m,k}(x_k^{(i)}) - a_i| < \frac{\epsilon}{3}\}$$

$$B_n = \{k \in : |t_{m,k}(x_k^{(n)}) - a_n| < \frac{\epsilon}{3}\}$$

Then $B_{ni}^c, B_i^c, B_n^c \in I$.
Let $B^c = B_{ni}^c \cap B_i^c \cap B_n^c$, where $B = \{k \in : |a_i - a_n| < \epsilon\}$.
Then $B^c \in I$.
We choose $k_0 \in B^c$,then for each $n, i \geq k_0$, we have

$$\{k \in : |a_i - a_n| < \epsilon\} \supseteq \{k \in : |t_{m,k}(x_k^{(i)}) - a_i| < \frac{\epsilon}{3}\}$$
$$\cap \{k \in : |t_{m,k}(x_k^{(n)}) - t_{m,k}(x_k^{(i)})| < \frac{\epsilon}{3}\}$$
$$\cap \{k \in : |t_{m,k}(x_k^{(n)}) - a_n| < \frac{\epsilon}{3}\}$$

Then (a_n) is a cauchy sequence of scalars in , so there exists a scalar $a \in$ such that $(a_n) \to a$, as $n \to \infty$.

(2) Let $0 < \delta < 1$ be given.Then we show that if $U = \{k \in: |t_{m,k}(x) - a| < \delta\}$,then $U^c \in I$.
Since $t_{m,k}(x^{(n)}) \to t_{m,k}(x)$,then there exists $q_0 \in$ such that

$$P = \{k \in: |t_{m,k}(x^{(q_0)} - t_{m,k}(x)| < \frac{\delta}{3}\} \qquad (1)$$

which implies that $P^c \in I$
The number q_0 can be so choosen that together with (1), we have

$$Q = \{k \in: |a_{q_0} - a| < \frac{\delta}{3}\}$$

such that $Q^c \in I$
Since $\{k \in: |t_{m,k}(x_k^{(q_0)}) - a_{q_0}| \geq \delta\} \in I$.Then we have a subset S of such that $S^c \in I$, where

$$S = \{k \in: |t_{m,k}(x_k^{(q_0)}) - a_{q_0}| < \frac{\delta}{3}\}.$$

Let$U^c = P^c \cap Q^c \cap S^c$, where $U = \{k \in: |t_{m,k}(x) - a| < \delta\}$.
Therefore for each $k \in U^c$, we have

$$\begin{aligned}\{k \in: |t_{m,k}(x) - a| < \delta\} \supseteq \{k \in: |t_{m,k}(x^{(q_0)} - t_{m,k}(x)| < \frac{\delta}{3}\} \\ \cap \{k \in: |t_{m,k}(x_k^{(q_0)}) - a_{q_0}| < \frac{\delta}{3}\} \\ \cap \{k \in: |a_{q_0} - a| < \frac{\delta}{3}\}\end{aligned}$$

Then the result follows.
Since the inclusions $V_{0\sigma}^I(m, \epsilon) \subset l_\infty$ and $V_\sigma^I(m, \epsilon) \subset l_\infty$ are strict so in view of Theorem 2.3 we have the following result.

Theorem 2.4. The spaces $V_{0\sigma}^I(m, \epsilon)$ and $V_\sigma^I(m, \epsilon)$ are nowhere dense subsets of l_∞.

Theorem 2.5. The space $V_{0\sigma}^I(m, \epsilon)$ is solid and monotone.

Proof. Let $(x_k) \in V_{0\sigma}^I(m, \epsilon)$ and α_k be a sequence of scalars with $|\alpha_k| \leq 1$, for all $k \in$
Then we have $|\alpha_k t_{m,k}(x)| \leq |\alpha_k||t_{m,k}(x)| \leq |t_{m,k}(x)|$, for all $k \in$
The space $V_{0\sigma}^I(m, \epsilon)$ is solid follows from the following inclusion relation.

$$\{k \in: |t_{m,k}(x)| \geq \epsilon\} \supseteq \{k \in: |\alpha_k t_{m,k}(x)| \geq \epsilon\}.$$

Also a sequence space is solid implies monotone. Hence the space $V^I_{0\sigma}(m,\epsilon)$is monotone.

Theorem 2.6. The inclusions $c^I_0 \subset V^I_{0\sigma}(m,\epsilon) \subset l_\infty$ are proper.

Proof. Let $x = (x_k) \in c^I_0$. Then we have $\{k \in: |x_k| \geq \epsilon\} \in I$
Since $c_0 \subset V_{0\sigma}(m,\epsilon)$
$x = (x_k) \in V^I_{0\sigma}$ implies $\{k \in: |t_{m,k}(x)| \geq \epsilon\} \in I$
Now let,

$$A_1 = \{k \in: |x_k| < \epsilon\} \in I$$

$$A_2 = \{k \in: |t_{m,k}(x)| < \epsilon\} \in I$$

be such that $A^c_1, A^c_2 \in I$.
As $l_\infty = \{x = (x_k) : \sup_k |x_k| < \infty\}$,taking supremum over k we get $A^c_1 \subset A^c_2$.
Hence $c^I_0 \subset V^I_{0\sigma}(m,\epsilon) \subset l_\infty$.

Theorem 2.7. The inclusions $c^I \subset V^I_\sigma(m,\epsilon) \subset l_\infty$ are proper.

Proof. Let $x = (x_k) \in c^I$. Then we have $\{k \in: |x_k - L| \geq \epsilon\} \in I$
Since $c \subset V_\sigma(m,\epsilon) \subset l_\infty$
$x = (x_k) \in V^I_\sigma(m,\epsilon)$ implies $\{k \in: |t_{m,k}(x) - L| \geq \epsilon\} \in I$
Now let,

$$B_1 = \{k \in: |x_k - L| < \epsilon\} \in I$$

$$B_2 = \{k \in: |t_{m,k}(x) - L| < \epsilon\} \in I$$

be such that $B^c_1, B^c_2 \in I$.
As $l_\infty = \{x = (x_k) : \sup_k |x_k| < \infty\}$,taking supremum over k we get $B^c_1 \subset B^c_2$.
Hence $c^I \subset V^I_\sigma(m,\epsilon) \subset l_\infty$.

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