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Fekete-Szegö Problem for Certain Subclass of Analytic Univalent Function using Quasi-Subordination

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Abstract

An analytic function f is quasi-subordinate to an analytic function g, in the open unit disk if there exist analytic functions φ and w, with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = \varphi(z)g(w(z))$. Certain subclass of analytic univalent functions associated with quasi-subordination are defined and the bounds for the Fekete-Szegö coefficient functional $|a_3 - \mu a_2^2|$ for functions belonging to these subclass is derived.

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1 Introduction and Motivation

Let $\mathcal{A}$ be the class of analytic function f in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For two analytic functions f and g, the function f is subordinate to g, written as follows:

$$f(z) \prec g(z), \tag{1}$$

if there exists an analytic function w, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. For brief survey on the concept of subordination, see [1].

Ma and Minda [2] introduced the following class

$$S^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}, \tag{2}$$

where ϕ is an analytic function with positive real part in $\mathbb{D}$, $\phi(\mathbb{D})$ is symmetric with respect to the real axis and starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in S^*(\phi)$ is called Ma-Minda starlike (with respect to ϕ). The class $C(\phi)$ is the class of functions $f \in \mathcal{A}$ for which $1 + zf''(z)/f'(z) \prec \phi(z)$. The class $S^*(\phi)$ and $C(\phi)$ include several well-known subclasses of starlike and convex functions as special case.

In the year 1970, Robertson [3] introduced the concept of quasi-subordination. For two analytic functions f and g, the function f is quasi-subordinate to g, written as follows:

$$f(z) \prec_q g(z), \tag{3}$$

if there exists analytic functions φ and w, with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = \varphi(z)g(w(z))$. Observe that when $\varphi(z) = 1$, then $f(z) = g(w(z))$, so that $f(z) \prec g(z)$ in $\mathbb{D}$. Also notice that if $w(z) = z$, then $f(z) = \varphi(z)g(z)$ and it is said that f is majorized by g and written $f(z) \ll g(z)$ in $\mathbb{D}$. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization. See [4, 5, 6] for works related to quasi-subordination.

Throughout this paper it is assumed that ϕ is analytic in $\mathbb{D}$ with $\phi(0) = 1$. Motivated by [2, 3], we define the following class.

Definition 1.1. *Let the class* $L_q(\lambda, \phi)$, $(0 \leq \lambda \leq 1)$, *consists of functions* $f \in \mathcal{A}$ *satisfying the quasi-subordination*

$$\frac{\lambda z^3 f''' + (1 + 2\lambda)z^2 f'' + zf'}{\lambda z^2 f'' + zf'} - 1 \prec_q \phi(z) - 1. \tag{4}$$

Example 1.2. *The function* $f : \mathbb{D} \to \mathbb{C}$ *defined by the following:*

$$\frac{\lambda z^3 f''' + (1 + 2\lambda)z^2 f'' + zf'}{\lambda z^2 f'' + zf'} - 1 = z(\phi(z) - 1) \tag{5}$$

belongs to the class $L_q(\lambda, \phi)$.

It is well known (see [10]) that the n-th coefficient of a univalent function $f \in \mathcal{A}$ is bounded by n. The bounds for coefficient give information about various geometric properties of the function. Many authors have also investigated the bounds for the Fekete-Szegö coefficient for various classes [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. In this paper, we obtain coefficient estimates for the functions in the above defined class.

Let Ω be the class of analytic functions w, normalized by $w(0) = 0$, and satisfying the condition $|w(z)| < 1$. We need the following lemma to prove our results.

Lemma 1.3. *(see [26]). If $w \in \Omega$, then for any complex number t*

$$|w_2 - tw_1^2| \leq \max\{1; |t|\}. \tag{6}$$

The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

2 Main Results

Throughout, let $f(z) = z + a_2z^2 + a_3z^3 + \ldots$, $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$, $\varphi(z) = c_0 + c_1z + c_2z^2 + c_3z^3 + \ldots$, $B_1 \in \mathbb{R}$ and $B_1 > 0$.

Theorem 2.1. *If $f \in \mathcal{A}$ belongs to $L_q(\lambda, \phi)$, $(0 \leq \lambda \leq 1)$, then*

$$\begin{aligned} |a_2| &\leq \frac{B_1}{2(1+\lambda)}, \\ |a_3| &\leq \frac{1}{6(1+2\lambda)}(B_1 + \max\{B_1, B_1^2 + |B_2|\}), \end{aligned} \tag{7}$$

and, for any complex number μ,

$$|a_3 - \mu a_2^2| \leq \frac{1}{6(1+2\lambda)}\left(B_1 + \max\left\{B_1, \left|1 - \frac{3(1+2\lambda)}{2(1+\lambda)^2}\mu\right| B_1^2 + |B_2|\right\}\right). \tag{8}$$

Proof. If $f \in L_q(\lambda, \phi)$, $(0 \leq \lambda \leq 1)$, then there exist analytic functions φ and w, with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{\lambda z^3 f''' + (1+2\lambda)z^2 f'' + zf'}{\lambda z^2 f'' + zf'} - 1 = \varphi(z)(\phi(w(z)) - 1). \tag{9}$$

Since

$$\frac{\lambda z^3 f''' + (1+2\lambda)z^2 f'' + zf'}{\lambda z^2 f'' + zf'} - 1 = 2(1+\lambda)a_2z + (-4(1+\lambda)^2a_2^2 + 6(1+2\lambda)a_3)z^2 + \cdots, \tag{10}$$

$$\phi(w(z)) - 1 = B_1w_1z + (B_1w_2 + B_2w_1^2)z^2 + \cdots,$$

$$\varphi(z)(\phi(w(z))-1)=B_1c_0w_1z+(B_1c_1w_1+c_0(B_1w_2+B_2w_1^2))z^2+\cdots, \quad (11)$$

it follows from (9) that

$$\begin{aligned} a_2 &= \frac{B_1c_0w_1}{2(1+\lambda)}, \\ a_3 &= \frac{1}{6(1+2\lambda)}(B_1c_1w_1+B_1c_0w_2+c_0(B_2+B_1^2c_0)w_1^2). \end{aligned} \quad (12)$$

Since $\varphi(z)$ is analytic and bounded in $\mathbb{D}$, we have [27, page 172]

$$|c_n|\leq 1-|c_0|^2\leq 1 \quad (n>0). \quad (13)$$

By using this fact and the well-known inequality, $|w_1|\leq 1$, we get

$$|a_2|\leq \frac{B_1}{2(1+\lambda)}. \quad (14)$$

Further,

$$a_3-\mu a_2^2=\frac{1}{6(1+2\lambda)}\left(B_1c_1w_1+c_0\left(B_1w_2+\left(B_2+B_1^2c_0-\frac{3(1+2\lambda)}{2(1+\lambda)^2}\mu B_1^2c_0\right)w_1^2\right)\right). \quad (15)$$

Then

$$|a_3-\mu a_2^2|\leq \frac{1}{6(1+2\lambda)}\left(|B_1c_1w_1|+\left|B_1c_0\left(w_2-\left(\frac{3(1+2\lambda)}{2(1+\lambda)^2}\mu B_1c_0-B_1c_0-\frac{B_2}{B_1}\right)w_1^2\right)\right|\right). \quad (16)$$

Again applying $|c_n|\leq 1$ and $|w_1|\leq 1$, we have

$$|a_3-\mu a_2^2|\leq \frac{B_1}{6(1+2\lambda)}\left(1+\left|w_2-\left(-\left(1-\frac{3(1+2\lambda)}{2(1+\lambda)^2}\mu\right)B_1c_0-\frac{B_2}{B_1}\right)w_1^2\right|\right). \quad (17)$$

Applying Lemma 1.3 to

$$\left|w_2-\left(-\left(1-\frac{3(1+2\lambda)}{2(1+\lambda)^2}\mu\right)B_1c_0-\frac{B_2}{B_1}\right)w_1^2\right| \quad (18)$$

yields

$$|a_3-\mu a_2^2|\leq \frac{B_1}{6(1+2\lambda)}\left(1+\max\left\{1,\left|-\left(1-\frac{3(1+2\lambda)}{2(1+\lambda)^2}\mu\right)B_1c_0-\frac{B_2}{B_1}\right|\right\}\right). \quad (19)$$

Observe that

$$\left|-\left(1-\frac{3(1+2\lambda)}{2(1+\lambda)^2}\mu\right)B_1c_0-\frac{B_2}{B_1}\right| \le B_1|c_0|\left|1-\frac{3(1+2\lambda)}{2(1+\lambda)^2}\mu\right|+\left|\frac{B_2}{B_1}\right|, \quad (20)$$

and hence we can conclude that

$$|a_3-\mu a_2^2| \le \frac{1}{6(1+2\lambda)}\left(B_1+\max\left\{B_1,\left|1-\frac{3(1+2\lambda)}{2(1+\lambda)^2}\mu\right|B_1^2+|B_2|\right\}\right). \quad (21)$$

For $\mu = 0$, the above will reduce to the estimate of $|a_3|$. □

Theorem 2.2. *If $f \in \mathcal{A}$ satisfies*

$$\frac{\lambda z^3 f''' + (1+2\lambda)z^2 f'' + zf'}{\lambda z^2 f'' + zf'} - 1 \ll \phi(z) - 1, \quad (22)$$

then the following inequalities hold:

$$|a_2| \le \frac{B_1}{2(1+\lambda)},$$
$$|a_3| \le \frac{1}{6(1+2\lambda)}(B_1+B_1^2+|B_2|), \quad (23)$$

and, for any complex number μ,

$$|a_3-\mu a_2^2| \le \frac{1}{6(1+2\lambda)}\left(B_1+\left|1-\frac{3(1+2\lambda)}{2(1+\lambda)^2}\mu\right|B_1^2+|B_2|\right). \quad (24)$$

Proof. The result follows by taking $w(z) = z$ in the proof of Theorem 2.1. □

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References

[1] P. Duren, Subordination, in Complex Analysis, Lecture Notes in Mathematics, Springer, Berlin, Germany, 599 (1977), 22–29.

[2] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in: Proceedings of the Conference on Complex Analysis, (Tianjin, 1992), Conference Proceedings Lecture Notes Analysis, International Press, Cambridge, Mass, USA, 1 (1994), 157–169.

[3] M.S. Robertson, Quasi-subordination and coefficient conjectures, Bulletin of the American Mathematical Soceity, 76 (1970), 1–9.

[4] O. Altintaş and S. Owa, Majorizations and quasi-subordinations for certain analytic functions, Proceedings of the Japan AcademyA, 68(7) (1992), 181–185.

[5] S.Y. Lee, Quasi-subordinate functions and coefficient conjectures, Journal of the Korean Mathematical Society, 12(1) (1975), 43–50.

[6] F.Y. Ren, S. Owa and S. Fukui, Some inequalities on quasi-subordinate functions, Bulletin of the Australian Mathematical Society, 43(2) (1991), 317–324.

[7] S.S. Miller, P.T. Mocanu and M.O. Reade, All α-convex functions are starlike, Revue Roumaine de Mathematiques Pures et Appliquees, 17 (1972), 1395–1397.

[8] Z. Lewandowski, S. Miller and E. Zlotkiewicz, Gamma-starlike functions, Annales Universitatis Mariae Curie-Sklodowska A, 28 (1976), 53–58.

[9] M. Darus and D.K. Thomas, α-logarithmically convex functions, Indian Journal of Pure and Applied Mathematics, 29(10) (1998), 1049–1059.

[10] L. de Branges, A proof of the Bieberbach conjecture, Acta Mathematica, 154(1-2) (1985), 137–152.

[11] H.R. Abdel-Gawad, On the Fekete-Szegö problem for alpha-quasi-convex functions, Tamkang Journal of Mathematics, 31(4) (2000), 251–255.

[12] O.P. Ahuja and M. Jahangiri, Fekete-Szegö problem for a unified class of analytic functions, Panamerican Mathematical Journal, 7(2) (1997), 67–78.

[13] R.M. Ali, V. Ravichandran and N. Seenivasagan, Coefficient bounds for p-valent functions, Applied Mathematics and Computation, 187(1) (2007), 35–46.

[14] R.M. Ali, S.K. Lee, V. Ravichandran and S. Supramaniam, The Fekete-Szegö coefficient functional for transforms of analytic functions, Bulletin of the Iranian Mathematical Society, 35(2) (2009), 119–142.

[15] N.E. Cho and S. Owa, On the Fekete-Szegö problem for strongly α-logarithmic quasiconvex functions, Southeast Asian Bulletin of Mathematics, 28(3) (2004), 421–430.

[16] J.H. Choi, Y.C. Kim and T. Sugawa, A general approach to the Fekete-Szegö problem, Journal of the Mathematical Society of Japan, 59(3) (2007), 707–727.

[17] M. Darus and N. Tuneski, On the Fekete-Szegö problem for generalized close-to-convex functions, International Mathematical Journal, 4(6) (2003), 561–568.

[18] M. Darus, T.N. Shanmugam and S. Sivasubramanian, Fekete-Szegö inequality for a certain class of analytic functions, Mathematica, 49(72)(1) (2007), 29–34.

[19] K.K. Dixit and S.K. Pal, On a class of univalent functions related to complex order, Indian Journal of Pure and Applied Mathematics, 26(9) (1995), 889–896.

[20] S. Kanas, An unified approach to the Fekete-Szegö problem, Applied Mathematics and Computation, 218 (2012), 8453–8461.

[21] S. Kanas and H.E. Darwish, Fekete-Szegö problem for starlike and convex functions of complex order, Applied Mathematics Letters, 23(7) (2010), 777–782.

[22] S. Kanas and A. Lecko, On the Fekete-Szegö problem and the domain of convexity for a certain class of univalent functions, Zeszyty Naukowe Politechniki Rzezowskiej. Matematyka i Fizyka, (10)(1990), 49–57.

[23] O.S. Kwon and N.E. Cho, On the Fekete-Szegö problem for certain analytic functions, Journal of the Korea Society of Mathematical Education B, 10(4) (2003), 265–271.

[24] V. Ravichandran, M. Darus, M.H. Khan and K.G. Subramanian, Fekete-Szegö inequality for certain class of analytic functions, The Australian Journal of Mathematical Analysis and Applications, 1(2) (2004), Article 2, 7 pages.

[25] V. Ravichandran, A. Gangadharan and M. Darus, Fekete-Szegö inequality for certain clas of Bazilevic functions, Far East Journal of Mathematical Sciences, 15(2) (2004) 171–180.

[26] F.R. Keogh and E.P. Merkes, A coefficient inquality for certain classes of analytic functions, Proc. Amer. Math. Soc., 20 (1969), 8–12.

[27] Z. Nehari, Conformal mapping, Dover, New York, NY, USA, 1975, Reprinting of the 1952 edition.

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