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Note to an Open Problem

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Abstract

In this paper, we give an answer to an open problem posed in the paper [Qinglong Huang, Improved Answers to an Open Problem Concerning an Integral Inequality, Mathematica Aeterna, Volume 2, 2012, number 4, 321-324].

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1 Introduction

In the paper [4], Quinglong Huang has posted the following open problem.

Open Problem 1.1 *Assume constant $\gamma > 0$. Let $f(x) \geq 0$ be a continuous function on $[0, 1]$ satisfying the inequality*

$$\int_t^1 f^\gamma(x)dx \geq \int_t^1 x^\gamma dx \quad \forall t \in [0, 1]. \tag{1}$$

Does the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx \tag{2}$$

hold for $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta < \gamma$.

We note that in [1] the following Theorem 1.2 was proved.

Theorem 1.2 *Let $f(x) \geq 0$ be a continuous function on $[0,1]$ satisfying*

$$\int_t^1 f(x)dx \geq \int_t^1 x dx \quad \forall t \in [0,1],$$

then

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx$$

holds for every real number $\alpha > 1$ and $\beta > 0$.

This theorem was an answer to the open problem posed in [2]. Improved answers to the problem were obtained by Quinglong Huang in [4].

2 Solution of an Open Problem

In this paper we give a negative answer to the Quinlog Huang's problem posed in [4]. We find two functions satisfying (1) for which the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx$$

is fulfilled for one function but is not fulfilled for another one.
Let $\gamma > 0$. Put $f_1(x) \equiv 1$, $x \in [0,1]$, then (1) can be written as

$$1 - t \geq \frac{1}{\gamma+1}(1 - t^{\gamma+1}).$$

Denote $g(t) = (1-t)(\gamma+1) - 1 + t^{\gamma+1}$ for $t \in [0,1]$. From $g(1) = 0$, $g'(t) = (\gamma+1)(t^\gamma - 1) < 0$ we have (1) is fulfilled for $t \in [0,1]$. The inequality (2) can be written as $1 \geq \frac{1}{\alpha+1}$, which is hold for all $0 < \alpha < \gamma$ and $\beta > 0$.

Now, we construct the second function.

Denote $s_0(t) = \frac{1}{\gamma+1}(1 - t^{\gamma+1})$ for $t \in [0,1]$. From $s_0'(t) = -t^\gamma < 0$, $s_0''(t) = -\gamma t^{\gamma-1} < 0$ it follows $s_0(t)$ is a decreasing concave function on $[0,1]$. Denote $t^\star = \frac{\gamma}{\gamma+1}$ then $A = [t^\star, 1 - t^\star]$ is the intersection point of three lines $p_1 : \quad y = 1 - t$, $p_2 : \quad y = \frac{1}{\gamma+1}$, $p_3 : \quad y = a - bt = \frac{1+\gamma b}{1+\gamma} - bt$, where $0 < b < 1$. Denote $A_1 = [t_1, y_1]$, $A_2 = [t_2, y_2]$ the intersection points of p_1, p_4 and p_2, p_4 lines, where $p_4 : \quad y = s_0(t^\star) + s_0'(t^\star)(t - t^\star)$. ($A_1$, A_2 exist, because of $p_1'(t) = -1$, $p_2'(t) = 0$, $p_4'(t) = s_0'(t^\star) = -t^{\star\gamma} \neq 0, \neq 1$). We show that

$0 < p_4(1) < \frac{1}{1+\gamma}$ and $\frac{1}{1+\gamma} < p_4(0) < 1$. It implies $t^\star < t_1 < 1$ and $t_0 < t_2 < t^\star$. Really.

$$p_4(1) = s_0(t^\star) + s_0'(t^\star)(1 - t^\star) = \frac{1}{1+\gamma}\left(1 - \left(\frac{\gamma}{1+\gamma}\right)^{1+\gamma} - \left(\frac{\gamma}{1+\gamma}\right)^{\gamma}\right) =$$

$$\frac{1}{1+\gamma}h(\gamma).$$

$h(\gamma) > 0$ is equivalent to $(2\gamma+1)\gamma^\gamma < (1+\gamma)^{1+\gamma}$ for $\gamma > 0$. Denote

$$H(\gamma) = \ln(2\gamma+1) + \gamma\ln(\gamma) - (1+\gamma)\ln(1+\gamma),$$

then $h(\gamma) > 0$ is equivalent to $H(\gamma) < 0$ for $\gamma > 0$. Elementary calculations give $H(0) = 0$, $\lim_{t\to 0^+} H'(t) = -\infty$,

$$H'(\gamma) = \frac{2}{2\gamma+1} + \ln(\gamma) - \ln(1+\gamma),$$

$$H''(\gamma) = \frac{-4}{(2\gamma+1)^2} + \frac{1}{\gamma(1+\gamma)} = \frac{1}{\gamma(1+\gamma)(2\gamma+1)^2} > 0.$$

It implies $H(\gamma) < 0$ for $\gamma > 0$, so $p_4(1) > 0$.
The second inequality $p_4(1) < \frac{1}{1+\gamma}h(\gamma)$ is evident. From

$$p_4(0) = s_0(t^\star) - t^\star s_0'(t^\star) = \frac{1}{1+\gamma}\left(1 - \left(\frac{\gamma}{1+\gamma}\right)^{1+\gamma}\right) + \left(\frac{\gamma}{1+\gamma}\right)^{1+\gamma} =$$

$$\frac{1}{1+\gamma} - \frac{\gamma^{1+\gamma}}{(1+\gamma)^{2+\gamma}} + \frac{\gamma^{1+\gamma}}{(1+\gamma)^{1+\gamma}}$$

we have $p_4(0) > \frac{1}{1+\gamma}$ is equivalent to $1+\gamma > 1$, which is evident. Similarly, $p_4(0) < 1$ is equivalent to $\gamma < 1+\gamma$.
Let $y = r_{\epsilon,b}(t)$ is a function given by

$$(t - t_3)^2 + (y - y_3)^2 = d^2([t_3, y_3], [t^\star - \epsilon, a - b(t^\star - \epsilon)]),$$

where

$$0 < \varepsilon < \min\left\{t^\star - t_2, t_1 - t^\star, \frac{(t_1 - t^\star)\sqrt{2}}{\sqrt{1+b^2}}\right\} = \min\{t^\star - t_2, t_1 - t^\star\} \qquad (3)$$

and $S = [t_3, y_3]$ is the intersection point of two lines $q_1: \quad y = 1 - 2t^\star - 2f + t$, $q_3: \quad y = a - (t^\star - \epsilon)(b + \frac{1}{b}) + \frac{1}{b}t$ (q_1, q_3 are perpendiculars to the lines p_1, p_3 in the points B_1, B_2), where $B_1 = [t^\star + f, 1 - t^\star - f]$, $B_2 = [t^\star - \epsilon, a - b(t^\star - \epsilon)]$,

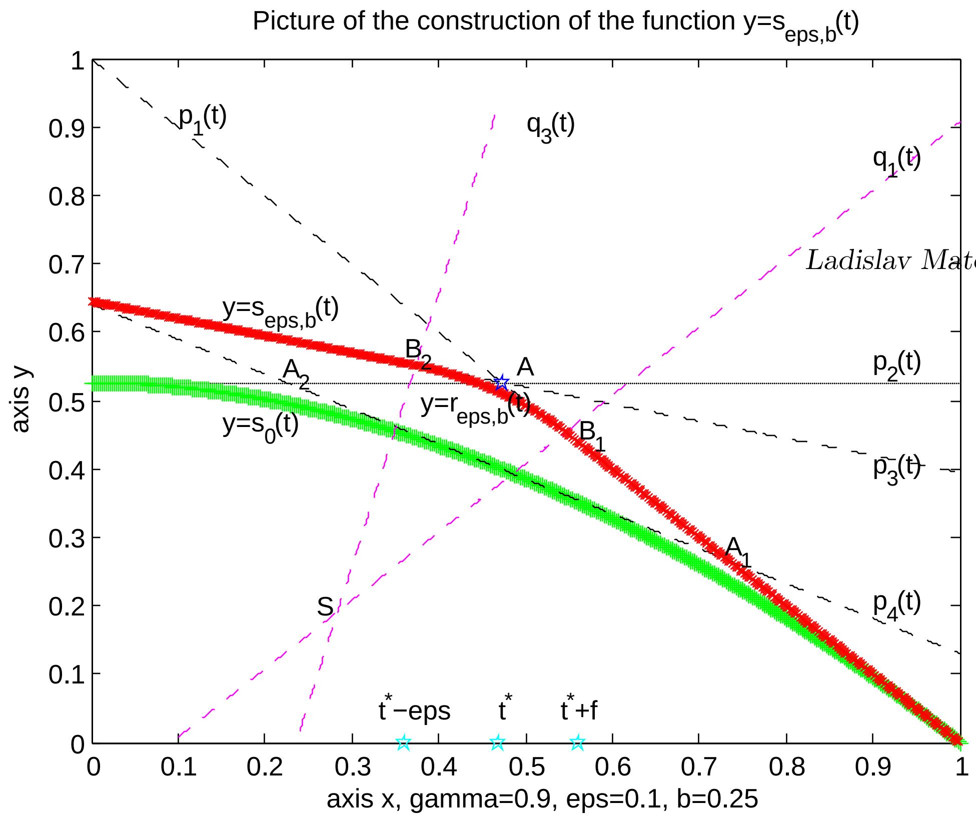

Figure 1:

$f = \frac{\sqrt{2}}{2}\epsilon\sqrt{1+b^2}$

$(d(A,B_1) = d(A,B_2),\ f = \sqrt{\frac{\epsilon^2+(a-b(t^\star-\epsilon))^2}{2}} = \frac{\sqrt{2}}{2}\epsilon\sqrt{1+b^2})$ and d is a distance of two points.

We note that S exist because p_1, p_3 are intersecting lines, and thus also q_1, q_3 are intersecting lines. From $B_1 \in p_1$ we have $s_0(t^\star+f) < 1-t^\star-f$. From $s_0(t) \leq \frac{1}{1+\gamma}$ and $p_3(t) > \frac{1}{1+\gamma}$ for $0 \leq t < t^\star$ we have $s_0(t^\star-\epsilon) < a-b(t^\star-\epsilon)$. From this we have $s_0(t) < r_{\epsilon,b}(t)$ for $t^\star-\epsilon \leq t \leq t^\star+f$ (the line segment B_1, B_2 is above the the line p_4 because of $y = s_0(t)$ is a concave function). It is easy to show that the function (Figure 1)

$$s_{\epsilon,b}(t) = \begin{cases} a-bt & \text{for } t \in [0, t^\star-\epsilon]; \\ r_{\epsilon,b}(t) & \text{for } t \in [t^\star-\epsilon, t^\star+f]; \\ 1-t & \text{for } t \in [t^\star+f, 1]; \end{cases}$$

is a continuous function with a continuous derivative such that $s'_{\epsilon,b}(t) < 0$, $s_{\epsilon,b}(1) = 0$, $s_{\epsilon,b}(0) = a$, $|s'_{\epsilon,b}(t)| \leq 1$ for $t \in [t^\star-\epsilon, t^\star+f]$, $s_{\epsilon,b}(t) \geq s_0(t)$ for $t \in [0,1]$. Now, we put

$$f^\star(t) = (-s'_{\epsilon,b}(t))^{\frac{1}{\gamma}} \quad \text{for} \quad t \in [0,1], \quad \gamma > 0. \tag{4}$$

It is evident that from $s_{\epsilon,b}(t) \geq s_0(t)$ for $t \in [0,1]$ we have that $f^\star(t)$ fulfils (1). Let $\gamma > 0$ and $0 < \lambda < \gamma$. We take $\epsilon < \frac{\gamma-\lambda}{3(1+\gamma)(1+\lambda)}$, satisfying (3) and b such that $0 < b < \left(\left(1+\frac{1}{\gamma}\right)\epsilon\right)^{\frac{\gamma}{\lambda}}$.

Then we have

$$\int_0^1 f^{\star\lambda}(t)dt = \int_0^{t^\star-\epsilon} (-s'_{\epsilon,b}(t))^{\frac{1}{\gamma}}dt + \int_{t^\star-\epsilon}^{t^\star+\epsilon} (-s'_{\epsilon,b}(t))^{\frac{1}{\gamma}}dt + \int_{t^\star+\epsilon}^{1} (-s'_{\epsilon,b}(t))^{\frac{1}{\gamma}}dt <$$

$$b^{\frac{\lambda}{\gamma}}t^\star + 2\epsilon + \frac{1}{\gamma+1} < 3\epsilon + \frac{1}{\gamma+1} < \frac{1}{\lambda+1}.$$

So

$$\int_0^1 f^{\star\lambda}(t)dt < \int_0^1 t^\lambda dt.$$

It is easy to show that

$$h(\beta) = g(f,\alpha,\beta) = \int_0^1 f^{\alpha+\beta}(t) - t^\alpha f^\beta(t)dt$$

is a continuous function for fixed $f = f^\star$ and $\alpha = \lambda > 0$. ($b^{\frac{1}{\gamma}} \le f^\star \le 1$.) From $h(0) = \int_0^1 f^\alpha(t) - t^\alpha dt < 0$ for $\alpha = \lambda$ and $f = f^\star$ there is $\beta_0 > 0$ such that $g(f^\star, \lambda, \beta) < 0$ for $0 \le \beta < \beta_0$. So, there is a function fulfilling (1) and $\beta_0(\lambda) > 0$, such that for arbitrary $\lambda, \beta > 0$, $\lambda < \gamma$, and $\beta < \beta_0(\lambda)$ the inequality (2) is not fulfilled.

Lastly we propose the following open problem.

3 Open Problem

What conditions have to be added to the condition (1), so that the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \ge \int_0^1 x^\alpha f^\beta(x)dx$$

was fulfilled for $\alpha \ge 0$, $\beta \ge 0$ and $\alpha + \beta < \gamma$.

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