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GEODESICS AND HELICES ON EUCLIDEAN SPACE

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Abstract

The relationships between the geodesic curves and helix curves on the hypercylinder on Euclidean space are given in [1]. The inclined curves on circular cylinder are called ordinary helices.

In this study, for the generalization of ordinary helices on hypercylinder and their relations to geodesic curves were given.

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1 Introduction

E. Müller defined helices as the curves which have constant angle with a fixed direction and named then as inclined curves [2].
Various studies are done on helices see for example [3], [4], [5]. Helices on cylinder in three dimensional are shown to be geodesics [1]. We investigate whether or not the same property holds in higher dimension. In this paper, we answer this question.

2 Preliminaries

An $(n-1)$-dimensional hypercylindir on n-dimensional Euclidean space E^n, is a point statement set as

$$C = \left\{ X = (x_1, x_2, ..., x_n) \mid x_i \in R, 1 \leq i \leq n, \sum_{i=1}^{n-1} x_i^2 = 1, x_n = k, k \in R \right\}$$

This cylinder is also denominated as $(n-1)$-cylinder [6].

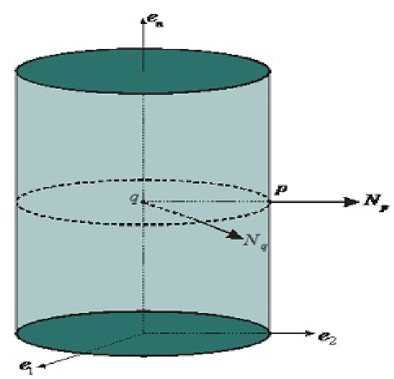

Figure 1: (n-1)-cylinder

C, the outer normals of the $(n-1)$-cylinder, can also be considered as the unit normal vector area on C (Figure 1).
Accordingly, the N vector area defined as $N_p = (p_1, p_2, ..., p_{n-1}, 0)$ for $P = (p_1, p_2, ..., p_n) \in C$ is the unit normal vector area of C. Besides,

$$\langle N, e_n \rangle = 0 \tag{1}$$

[7].

Let the unit tangent vector area of the curve $M \subset E^n$ be V_1 and $X \in \chi(E^n)$ be the constant unit vector area. If for $P \in M$

$$\langle V_1, X \rangle |_p = \cos\varphi = constant, \varphi \neq \frac{\pi}{2}$$

then the curve M is called an inclined curves on E^n, the angle φ is called the incline angle of M and the space $Sp\{X\}$ is called the incline axis of M. If the condition $\varphi \neq \frac{\pi}{2}$ is cancelled, each curve on E^n becomes an inclined curves on E^{n+1} [7].

If Y is a c^∞ vector area on a curve $\alpha : I \longrightarrow E^n$ and $D_T Y = 0$ on α , then the vector space Y is called a parallel vector area on the curve α . If $D_T Y = 0$ on a curve α , then the curve α is called a geodesic curve [7].

3 A THEOREM FOR GEODESICS and HELICES ON EUCLIDEAN SPACE

Theorem 3.1. *Let there be a circular cylinder*

$$C = \Big\{ X = (x_1, x_2, x_3) \in E^3 \mid x_1^2 + x_2^2 = 1, x_3 = k, k \in R \Big\}$$

on the 3-dimensional Euclidean space E^3. For a curve $\alpha : I \longrightarrow C$ on C to be geodesic, the required and sufficient condition is that the curve α is an inclined curves on C [1].

Proof. Let a curve $\alpha : I \longrightarrow C$ on a circular cylinder C be a geodesic curve. Given that the arc parameter of the curve α is t,

$$\alpha' = \frac{d\alpha}{dt} = V_1.$$

And if the angle between V_1 and $\frac{\partial}{\partial x_3}$ is $\varphi(t)$ for every t then

$$\langle V_1, \frac{\partial}{\partial x_3} \rangle = \cos \varphi(t).$$

Here, with the covariant derivative according to V_1,

$$\langle D_{V_1} V_1, \frac{\partial}{\partial x_3} \rangle + \langle V_1, D_{V_1} \frac{\partial}{\partial x_3} \rangle = -\sin \varphi(t) \frac{d\varphi}{dt}$$

or

$$\langle k_1 V_2, \frac{\partial}{\partial x_3} \rangle = -\sin \varphi(t) \frac{d\varphi}{dt}. \tag{2}$$

Since [7]

$$V_2 = \frac{\alpha''}{\|\alpha''\|}, \|\alpha''\| = k_1$$

then the expression (2) is

$$\langle \alpha'', \frac{\partial}{\partial x_3} \rangle = -\sin \varphi(t) \frac{d\varphi}{dt}. \tag{3}$$

Since unit velocity curve α is a geodesic, we get

$$\alpha'' = \lambda N$$

and for the cylinder C, using (1),

$$\langle N, \frac{\partial}{\partial x_3} \rangle = 0.$$

Thus, (3) is

$$\sin\varphi(t)\frac{d\varphi}{dt} = 0.$$

And therefore,

$$\sin\varphi(t) = 0$$

or

$$\frac{d\varphi}{dt} = 0.$$

And so,

$$\varphi(t) = 0$$

or

$$\varphi(t) = constant.$$

In that case, the curve α is an inclined curves with an axis of $\frac{\partial}{\partial x_3}$ on the circular cylinder C.
Contrarily, let the curve $\alpha : I \longrightarrow C$ be an inclined curves on C. If

$$C = \left\{X = (x_1, x_2, x_3) \in E^3 \mid x_1^2 + x_2^2 = 1, x_3 = k, k \in R\right\},$$

is a circular cylinder on E^3 then the axis of this cylinder is $\frac{\partial}{\partial x_3}$. Let a curve $\alpha : I \longrightarrow C$ be an inclined curves with an axis of $\frac{\partial}{\partial x_3}$ on C. Given that the parameter of α (arc parameter) is t,

$$\langle V_1, \frac{\partial}{\partial x_3}\rangle = \cos\varphi(t), \varphi(t) \neq \frac{\pi}{2}, (\varphi = constant).$$

Here we get

$$\langle k_1 V_2, \frac{\partial}{\partial x_3}\rangle = 0$$

where covariant derivative according to V_1 is

$$\langle \frac{dV_1}{dt}, \frac{\partial}{\partial x_3}\rangle = 0, k_1 \neq 0.$$

So, we can get

$$\langle V_2, \frac{\partial}{\partial x_3}\rangle = 0, V_2 = \frac{d^2\alpha}{dt^2} = \alpha''$$

or

$$\langle \alpha'', \frac{\partial}{\partial x_3}\rangle = 0$$

and then

$$\langle N, \frac{\partial}{\partial x_3}\rangle = 0$$

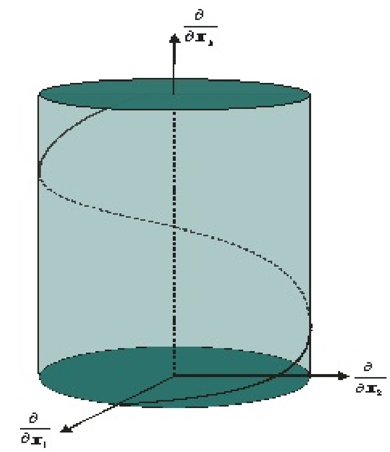

Figure 2: Circular helix

using (1) (Figure 2). On the other part,

$$\langle \alpha'', \alpha' \rangle = 0$$

and

$$\langle N, \alpha' \rangle = 0.$$

In this case, using Figure 3, if

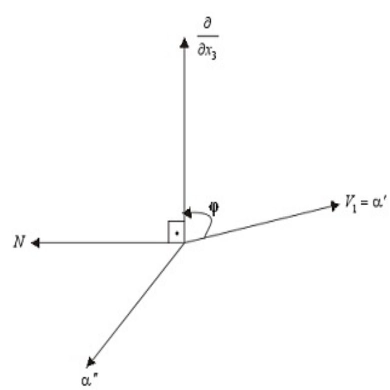

Figure 3: Frenet frame and others

$$N = \lambda \frac{\partial}{\partial x_3} \wedge \alpha',$$

$$\alpha'' = \mu \frac{\partial}{\partial x_3} \wedge \alpha'$$

then

$$\alpha'' = \mu N.$$

That is, the inclined curves α is a geodesic.

Theorem 3.2. *Given an (n-1)-hypercylinder*

$$C = \left\{ X = (x_1, x_2, ..., x_n) \mid x_i \in R, 1 \leq i \leq n, \sum_{i=1}^{n-1} x_i^2 = 1, x_n = k, k \in R \right\}$$

on n-dimensional Euclidean space E^n, if a curve $\alpha : I \to C$ on C is geodesic, then the curve α is an inclined curves on C.

Proof. $\alpha' = \frac{d\alpha}{dt} = V_1$ where the arc parameter of the curve α is t.
If the angle between V_1 and $\frac{\partial}{\partial x_n}$ is $\varphi(t)$ for every t then

$$\langle V_1, \frac{\partial}{\partial x_n} \rangle = \cos \varphi(t).$$

Here, with the covariant derivative according to V_1,

$$\langle D_{V_1} V_1, \frac{\partial}{\partial x_n} \rangle + \langle V_1, D_{V_1} \frac{\partial}{\partial x_n} \rangle = -\sin \varphi(t) \frac{d\varphi}{dt}$$
$$\langle k_1 V_2, \frac{\partial}{\partial x_n} \rangle = -\sin \varphi(t) \frac{d\varphi}{dt}. \tag{4}$$

Since

$$V_2 = \frac{\alpha''}{\|\alpha''\|}, \|\alpha''\| = k_1$$

then the statement (4) is

$$\langle \alpha'', \frac{\partial}{\partial x_n} \rangle = -\sin \varphi(t) \frac{d\varphi}{dt}. \tag{5}$$

Since α unit velocity curve α is a geodesic, we get

$$\alpha'' = \lambda N$$

and for the cylinder C, using (1),

$$\langle N, \frac{\partial}{\partial x_n} \rangle = 0.$$

Thus, the statement (5) is

$$\sin \varphi(t) \frac{d\varphi}{dt} = 0$$

and this is

$$\sin \varphi(t) = 0$$

or

$$\frac{d\varphi}{dt} = 0.$$

So,

$$\varphi(t) = 0 \text{ or } \varphi(t) = constant.$$

In that case, the curve α is an inclined curves with an axis of $\frac{\partial}{\partial x_n}$ on C (n-1)-cylinder.

Corollary 3.3. *The geodesic curves on the (n-1)-dimensional hypercylinder on n-dimensional Euclidean space E^n are inclined curves.*

Note that, the opposite of Theorem 1 is not always true. Indeed, for the inclined curves α,

$$N \in Sp\left\{\alpha', \frac{\partial}{\partial x_n}\right\}^{\perp}$$

and

$$\alpha'' \in Sp\left\{\alpha', \frac{\partial}{\partial x_n}\right\}^{\perp} (Figure 4).$$

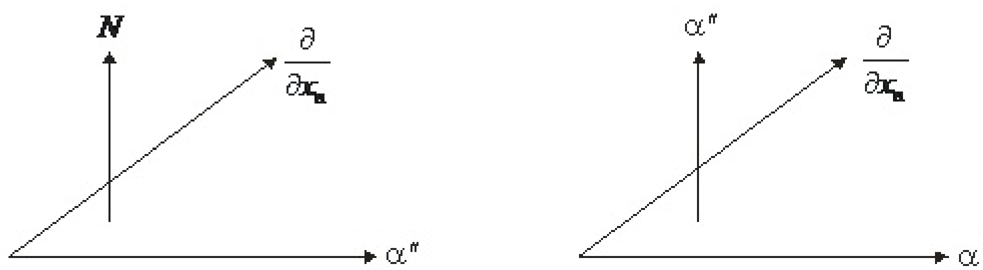

Figure 4: The vectors $\alpha', \alpha'', \frac{\partial}{\partial x_n}$ and N

However, for α'' and N to be linearly dependent, the dimension of the orthogonal space should be 1.

$$boySp\left\{\alpha', \frac{\partial}{\partial x_n}\right\}^{\perp} = n - 2$$

However, for $n = 3$, it has to be

$$boySp\left\{\alpha', \frac{\partial}{\partial x_3}\right\}^{\perp} = 1.$$

Therefore,

$$\alpha'' = \lambda N.$$

Corollary 3.4. *An inclined curves on (n-1)-dimensional hypercylinder on n-dimensional Euclidean space E^n are not geodesic curves.*

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