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ON THE STABILITY OF THE FUNCTIONAL EQUATION

$$f(x+y+z+xy+yz+xz+xyz) = f(x)+f(y)+f(z)+(x+y+xy)f(z)$$
$$+(y+z+yz)f(x)+(x+z+xz)f(y)$$

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Abstract

In this paper, we study the Hyers - Ulam stability and the Superstability of the functional equation

$$\begin{aligned} &f(x+y+z+xy+yz+xz+xyz) \\ &\quad = f(x)+f(y)+f(z)+(x+y+xy)f(z) \\ &\qquad + (y+z+yz)f(x)+(x+z+xz)f(y). \end{aligned}$$

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1 Introduction

In 1940, S.M.Ulam [20] while he was giving a series of lectures in the University of Wisconsin; he raised a question concerning the stability of homomorphism.

Let G_1 be a group and let G_2 be a metric group with the metric $d(.,.)$. Given $\varepsilon > 0$ does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$. Then a homomorphism $H : G_1 \to G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The first partial solution to Ulam's question was provided by D.H. Hyers [6]. Indeed, he proved the following celebrated theorem.
Theorem (D.H. Hyers): Assume that X and Y are Banach spaces. If a function $f : X \to Y$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \tag{1}$$

for some $\varepsilon \geq 0$ and for all $x\ inX$, then the limit

$$a(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exist for each x in X and $a : X \to Y$ is the unique additive function such that

$$\|f(x) - a(x)\| \leq \varepsilon$$

for any $x \in X$, moreover, if $f(tx)$ is continuous in t for each filed $x \in E$, then a is linear.

From the above case, we say that the additive functional equation $f(x+y) = f(x) + f(y)$ has the Hyers-Ulam stability on (X, Y). D.H. Hyers explicity constructed the additive function $a : X \to Y$ directly from the given function f. This method is called a direct method and it is a powerful tool for studying stability of functional equations.

Th.M.Rassias [15] proved the following substantial generalization of the result of Hyers:

Theorem 1.1 *Let X and Y be Banach spaces, let $\theta \in [0, \infty)$, and let $P \in [0, 1)$. If a functional equation $f : X \to Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \left(\|x\|^p + \|y\|^p \right)$$

for all $x, y \in X$, then there is a unique additive mapping $A : X \to Y$

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p} \|f(x)\|^p$$

for all $x \in X$. If in addition ,$f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

Due to this fact, the cauchy functional equation $f(x + y) = f(x) + f(y)$ is said to have the Hyers - Ulam - Rassias stability properly on (X, Y). A number of result concerning stability of different equations can be found in $[1, 2, 3, 5, 8]$. Consider the following functional equations

$$f(xy) = xf(y) + yf(x) \tag{2}$$

and

$$f(x^2) = 2xf(x) \tag{3}$$

which define multiplicative derivations and multiplicative Jordan derivations in algebras. It may be observed that real - valued function $f(x) = xlogx$ be a solution of the functional equation (2) and (3) on the interval $(0, \infty)$. Extend the result of (2), we obtain

$$f(xyz) = xyf(z) + yzf(x) + xzf(y). \tag{4}$$

During the 34th international Symposium on Functional Equations, Gy. Maksa [13] posed the Hyers - Ulam Stability problem for the functional equation (2) on the interval $(0, 1]$. The first result concerning the superstability of this equation for functions between operator algebras was obtained by P. Semrl [16]. On the other hand, Zs. Pales [14] remarked that the functional equation (2) for real - valued functions on $[1, \infty)$ is stable in the sense of Hyers and Ulam. The Hyers - Ulam Stability of the functional equations

$$h(rx^2 + 2x) = 2rxh(x) + 2h(x) \tag{5}$$

and

$$h(x + y + rxy) = h(x) + h(y) + rxh(y) + ryh(x) \tag{6}$$

were invested by E.H.Lee, I.S. Chang and Y.S. Chang [12] relative to a Multiplicative derivation.

A generalized version of the Hyers Ulam Stability and Superstability of the functional Equations

$$f(x + y - xy) + xf(y) + yf(x) = f(x) + f(y) \tag{7}$$

was investigated by Y.S. Jung [10].

In this paper, we study the Hyers-Ulam Stability and Superstability of the functional equation

$$\begin{aligned} f(x + y + z + xy + yz + xz + xyz) \\ = f(x) + f(y) + f(z) + (x + y + xy)f(z) + (y + z + yz)f(x) \\ + (x + z + xz)f(y). \end{aligned} \tag{8}$$

Throughout this paper, let N denote the set of all natural numbers and R denote the set of all real numbers.

2 Solutions of Equation(8)

In this section, we try to get the general solution of the functional equation (8) in the interval $(-1, \infty)$. Note that the function, $f(x) = (x + 1)ln(x + 1)$ is the solution of the functional equation (8) on the interval $(-1, \infty)$.

Theorem 2.1 *Let X be a real (or complex) linear space. A function $f : (-1,\infty) \to X$ satisfies the functional equation (8) for all $x \in (-1,\infty)$ if and only if there exists a solution $D : (0,\infty) \to X$ of the functional equation (4) such that $f(x) = D(x+1)$ for all $x \in (-1,\infty)$.*

Proof. Necessity. Define a mapping $D : (0,\infty) \to X$ by $D(x) := f(x-1)$. We claim that D is a the solution of the functional equation(4). Indeed,for all $x, y \in (0,\infty)$, we have

$$\begin{aligned} D(xyz) &= f(xyz-1) \\ &= f((x-1)+(y-1)+(z-1)+(x-1)(y-1)+(y-1)(z-1) \\ &\quad + (x-1)(z-1)+(x-1)(y-1)(z-1)) \\ &= xyD(z)+yzD(x)+xzD(y). \end{aligned}$$

Hence D is a solution of the functional equation(4). From the definition of D , we obtain $f(x) = D(x+1)$ for all $x \in (-1,\infty)$. The sufficiency part is obvious.

3 Hyers - Ulam stability of Equation(8)

In the following Theorem, we state the result due to F.Skof [17] which is concerning the stability of the additive functional equation $f(x+y) = f(x)+f(y)$ on a restricted domain.

Theorem 3.1 *Let X be a real (or complex) Banach space.Given $c > 0$, let a mapping f : [0,c)$\to X$ satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \delta$$

for some $\delta \ge 0$ and for all $x, y \in [0,c)$ with $x+y \in [0,c)$. Then there exists additive mapping $A : R \to X$ such that

$$\|f(x) - A(x)\| \le 3\delta$$

for all $x \in [0,c)$.

We now present our main theorem on the the Hyers - Ulam stability on the interval $(-1,0]$ of the functional equation (8). The proof is similar to the one given in [19].

Theorem 3.2 *Let X be a real (or complex) Banach space, and let $f : (-1,0] \to X$ be a mapping statisfying the inquality*

$$\Big\| f(x+y+z+xy+yz+xz+xyz) - f(x) - f(y) - f(z) \\ - (x+y+xy)f(z) - (y+z+yz)f(x) - (x+z+xz)f(y) \Big\| \le \delta \quad (9)$$

for some $\delta > 0$ *and for all* $x, y \in (-1, 0]$. *Then there exists a solution* $H : (-1, 0) \to X$ *of the functional equation(8) such that*

$$\|f(x) - H(x)\| \leq (4e)\delta \tag{10}$$

for all $x, y \in (-1, 0]$.

Proof. Let $g : (-1, 0] \to X$ be a mapping defined by

$$g(x) = \frac{f(x)}{x+1}$$

for all $x \in (-1, 0]$. Then, by (8), we observe that g statisfies inequality

$$\|g(x + y + z + xy + yz + xz) - g(x) - g(y) - g(z)\| \leq \frac{\delta}{(x+1)(y+1)(z+1)}$$

for all $x, y \in (-1, 0)$. Let us now define the mapping $F : [0, \infty) \to X$ by

$$F(-ln(x+1)) = g(x)$$

for all $x \in (-1, 0]$, then, by setting $u = -ln(x+1)$, $v = -ln(y+1)$ and $w = -ln(z+1)$, it will lead to

$$\|F(u + v + w) - F(u) - F(v) - F(w)\| \leq \delta e^{u+v+w} \tag{11}$$

for all $u, v, w \in (0, \infty]$. This means that

$$\|F(u + v + w) - F(u) - F(v) - F(w)\| \leq \delta e^{c} \tag{12}$$

for $u, v, w \in [0, c)$ with $u + v + w < c$, where $c > 1$ is an arbitrary given constant.

By using Theorem (9), we see that there exists an additive mapping $A : R \to X$ such that $\|F(u) - A(u)\| \leq 3\delta e^c$, for all $u \in [0, c)$. If we let $c \to 1$ in the last inequality, then we obtain

$$\|F(u) - A(u)\| \leq 3e\delta \tag{13}$$

for all $u \in [0, 1]$. Moreover, from (11)it follows

$$\begin{array}{c}
\|F(u+2) - F(u) - 2F(1)\| \leq \delta e^{u+2} \\
\|F(u+4) - F(u+2) - 2F(1)\| \leq \delta e^{u+4} \\
. \qquad . \qquad . \\
. \qquad . \qquad . \\
. \qquad . \qquad . \\
\|F(u+2k) - F(u+2k-2) - 2F(1)\| \leq \delta e^{u+2k}
\end{array}$$

for all $u \in [0,1]$ and $k \in N$. Summing up the above inequalities, we obtain $\|F(u+2k) - F(u) - 2kF(1)\| \le \delta e.e^{u+2k}(1 + e^{-2} + e^{-4} + ... + e^{-2k+2})$

$$\|F(u+2k) - F(u) - 2kF(1)\| \le \delta e.e^{u+2k} \tag{14}$$

for all $u \in [0,1]$ and $k \in N$. From equation (13), we assert that

$$\|F(v) - A(v)\| \le 4\delta e.e^{v} \tag{15}$$

for all $v \in [0,\infty)$.

Infact, when $v \ge 0$ and $k \in NU\{0\}$, we arrive that $v - k \in [0,1]$. Then by (13) and (14), we have

$$\begin{aligned}\|F(v) - A(v)\| &\le \|F(v) - F(v-2k) - 2kF(1)\| \\ &\quad + \|F(v-2k) - A(v-2k)\| + \|A(2k) - 2kF(1)\| \\ &\le \delta e.e^{v} + 3\delta e + 2k\,\|A(1) - F(1)\| \\ &\le \delta e.e^{v} + 3\delta e + 3\delta e.v \\ &\le \delta e(e^{v} + 3(1+v)) \\ &\le 4\delta e.e^{v}.\end{aligned}$$

Hence, from (15) and using the definition of F, it follows that

$$\begin{aligned}\|g(x) - A(-ln(x+1)\| &\le 4\delta e.e^{-ln(x+)} \\ &= \frac{4\delta e}{x+1}\end{aligned}$$

for all $x \in (-1,0]$. Again using the definition of $f(x)$, we obtain

$$\left\|\frac{f(x)}{x+1} - A(-ln(x+1)\right\| \le \frac{4\delta e}{x+1} \tag{16}$$

for all $x \in (-1,0]$. If we put $H(x) = (x+1)A(-ln(x+1))$ for all $x \in (-1,0]$, using Theorem (2.1) it can be easily verified that H is ia solution of the functional equation (8). Using $H(x)$ and equation (16) it will yield that

$$\|f(x) - H(x)\| \le (4e)\delta$$

for all $x \in (-1,0]$. This proves the equation (10). Hence the proof of the theorem is complete.

4 Superstability of Equation(8)

In this section, we will introduce the following Theorem (4.1) due to F.skof [18] which is esential to prove the main Theorem.

Theorem 4.1 *Let X be a real (or complex) Banach space, and let $c > 0$ be a given constant. Suppose that a mapping $f : R \to X$ statisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for some $\delta \geq 0$ and for all $x, y \in R$ with $|x| + |y| > c$. Then there exists a unique additive mapping $A : R \to X$ such that

$$\|f(x) - A(x)\| \leq 9\delta$$

for all $x \in R$.

Now let us prove the main theorem of section which is the super stability of the functional equation (8) on the interval $[0, \infty)$.

Theorem 4.2 *Let X be a real (or complex) Banach space, and let $f : [0, \infty) \to X$ be a mapping statisfying the inequality*

$$\begin{aligned} \Big\| f(x+y+z+xy+yz+xz+xyz) - f(x) - f(y) - f(z) \\ - (x+y+xy)f(z) - (y+z+yz)f(x) - (x+z+xz)f(y) \Big\| \leq \delta \quad (17) \end{aligned}$$

for some $\delta > 0$ and for all $x, y \in [0, \infty)$. Then f statisfies the functional equation (8) for all $x, y \in [0, \infty)$.

Proof. Defining the mapping $g : [0, \infty) \to X$ by $g(x) = \frac{f(x)}{x+1}$ for all $x \in [0, \infty)$ as in the proof of Theorem (3.2) and define the mapping $F : [0, \infty) \to X$ by $F(ln(x+1) = g(x)$ for all $x \in [0, \infty)$. Taking $u = ln(x+1)$, $v = ln(y+1)$, and $w = ln(z+1)$, we have

$$\|F(u+v+w) - F(u) - F(v) - F(w)\| \leq \delta e^{-(u+v+w)} \quad (18)$$

for all $u, v, w \in [0, \infty)$. From this, we claim that F is additive. From (18) with $\delta_n = \delta e^{-n} (n \in N)$, it gives $\|F(u+v+w) - F(u) - F(v) - F(w)\| \leq \delta_n$ for all $u, v, w \in [0, \infty)$ with $u+v+w > n$.

Now define a mapping $T : R \to X$ by

$$T(u) = \begin{cases} F(u) & \text{for } u \geq 0 \\ -F(-u) & \text{for } u < 0. \end{cases}$$

From this, we observe that

$$\|T(u+v) - T(u) - T(v)\| \leq \delta_n$$

for all $u, v \in R$ with $|u| + |v| > n$. Therefore, by Theorem 4.1, there exists a unique additive mapping $A_n : R \to X$, such that

$$\|T(u) - A_n(u)\| \leq 9\delta_n \quad (19)$$

for all $u \in R$. Let $m, n \in N$ with $n > m$. Then the additive mapping $A_n : R \to X$ statisfies $\|T(u) - A_n(u)\| \leq 9\delta_m$ for all $u \in R$. The uniqueness argument now implies $A_n = A_m$ for all $n \in N$ with $n > m > 0$, and thus $A_1 = A_2 = ... = A_n = ...$ Taking the limit in (19) as $n \to \infty$, it gives $T = A_n = A_1$ and this shows that F is additive.

Now, according to the definitions of F and g, we have $\frac{f(x)}{x+1} = F(ln(x+1))$ for all $x \in [0, \infty)$, and hence by using Theorem (2.1) we see that f statisfies the functional equation (8) for all $x, y \in [0, \infty)$. Since F is additive and $D(x) = xF(ln(x))$ $(x \in [1, \infty))$ is a solution of the functional equation (4). This is completes the proof of the theorem.

5 Generalized version of the Hyers-Ulam Stability of Equation(8)

In this section, we are going to investigate a generlized version of the Hyers-Ulam Stability of the followed equation (8) on the interval $[0, 1)$. In order to prove our main Theorem, we need the following definition and proposition which are proved by J. Tabor [19] concerning the stability of the additive functional equation $f(x + y) = f(x) + f(y)$ on the interval $[0, \infty)$.

Definition. A function $g : [0, \infty) \to [0, \infty)$ is called exponentially increasing if it is increasing and there exists $\gamma > 1$ and $h \in [0, \infty)$ such that $g(x + h) \geq \gamma g(x)$. for all $x \in [0, \infty)$.

Proposition 5.1. *Suppose that* $g : [0, \infty) \to [0, \infty)$ *is exponentially increasing with constants* γ *and* h *as in Definition, and that* $g(0) > 0$.
Let $K = 2\frac{g(h)}{g(0)} + \frac{\gamma}{\gamma - 1}$, *and let* $f : [0, \infty) \to X$ *be an arbitrary function such that*

$$f(x + y) - f(x) - f(y) \in g(x + y)V$$

for all $x \in 0, \infty)$. *Then there exists a unique additive function* $A : [0, \infty) \to X$ *such that* $A(h) = f(h)$ *and that*

$$f(x) - A(x) \in Kg(x)V$$

for all $x \in [0, \infty)$.

Throughout this section, we assume that X is a sequentially complete topological vector space and V is a closed convex, bounded and symmetric with respect to zero subset of X. The proof of the following Theorem is very analogous to one given in [19].

Theorem 5.1 *Let* $f : [0, 1) \to X$ *be a function such that*

$$\begin{aligned} &f(x + y + z + xy + yz + xz + xyz) - f(x) - f(y) - f(z) \\ &\quad - (x + y + xy)f(z) - (y + z + yz)f(x) - (x + z + xz)f(y) \in V \quad (20) \end{aligned}$$

for all $x, y \in [0,1)$, and let $z \in (0,1)$ be an arbitrary fixed. Then there exists a unique function $F_z : [0,1) \to X$ such that

$$F_z(z) = f(z) \tag{21}$$

$$F_z(x+y+z+xy+yz+xz+xyz) - (x+y+xy)F_z(z) \\ - (y+z+yz)F_z(x) - (x+z+xz)F_z(y) = F_z(x) + F_z(y) + F_z(z) \tag{22}$$

and that

$$f(x) - F_z(x) \in K_z V \tag{23}$$

for all $x, y \in [0,1)$, where $K_z = \frac{2}{1+z} + \frac{1}{z}$.

Proof. Let K be a set of real numbers. By X^K we denote the vector space of all functions from K into X. We define the linear operator $B : X^{[0,1)} \to X^{[0,\infty)}$ by the formula $B(f)(x) = exp(x)f(1+exp(-x))$ for all $x \in [0,\infty)$. Now from the equation (20), we can show that f also satifies the following equation

$$B(f)(u+v+w) - B(f)(u) - B(f)(v) - B(f)(w) \in exp(u+v+w)V$$

for $u, v, w \in [0,\infty)$ and so they are equivalent. Obviously exp is exponentially increasing with
$h := -exp^{-1}(1+z) = -ln(1+z) =, \gamma := exp(h) = \frac{1}{1+z}$. Therefore by Proposition 5.1, there exists a unique

$$A_h(h) = B(f)(h) \tag{24}$$

$$A_h(u+v+w) = A_h(u) + A_h(v) + A_h(w) \tag{25}$$

$$B(f)(u) - A_h(u) \in K_z exp(u)V \tag{26}$$

for all $x \in [0,\infty)$, where $K_z = 2\frac{exp(h)}{1+z} + \frac{\gamma}{\gamma-1} = \frac{2}{1+z} + \frac{1}{z}$.

Let $F_z := B^{-1}(A_h)$. Then we can easily verify from (24), (25) and (26) that F_z satisfies (21), (22) and (23), respectively.

Now we claim that F_z is unique. Suppose that there exists $F_z^{'}$ satisfying (24), (25) and (26). Then $B(F_z^{'})$ satifies (21), (22) and (23), hence $B(F_z^{'}) = A_h = B(F_z)$. Since B is bijection, this implies that $F_z^{'} = F_z$. Hence the proof of the theorem is complete.

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