Mathematica Aeterna, Vol. 3, 2013, no. 4, 241 - 244

Inequalities for the incomplete beta function

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Abstract

In this paper, we prersent some inequalities involving the incomplete beta function.

Mathematics Subject Classification: 26D15

Keywords: Incomplete beta function, inequality

1 Introduction

The beta function is defined by

$$\beta(a,b)=\int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}dt,$$

where $a,b>0$.

The incomplete beta function is defined by

$$\beta(a,b,x)=\int_x^\infty \frac{t^{a-1}}{(1+t)^{a+b}}dt,$$

where $a,b,x>0$. We let $\beta(a,b,0)=\beta(a,b)$.

In 2010, Sulaiman [1] gave the inequalities as follows.

$$\beta(a,b,x)\beta(a,b,y)\geq\beta(a,b,xy)\beta(a,b,1) \tag{1}$$

where $a,b>0$ and $x,y>1$.

$$\beta(a,b,x)\beta(a,b,y)\leq\beta(a,b,xy)\beta(a,b,1) \tag{2}$$

where $a,b>0$ and $0<y<1<x$.

$$\beta(a,b,x)\beta(a,b,y) \leq \beta(a,b,x+y)\beta(a,b,0) \tag{3}$$

where $0 < a < 1$ and $b, x, y > 0$.

In this paper, we present the generalizations for the inequalities (1), (2) and (3).

2 Results

Theorem 2.1. *Let $a, b, c > 0$ and $x, y > c$. Then*

$$\beta(a,b,x)\beta(a,b,y) \geq \beta(a,b,\frac{xy}{c})\beta(a,b,c). \tag{4}$$

Proof. Let $g(t) = \dfrac{t^{a-1}}{(1+t)^{a+b}}$, $F(t) = \dfrac{\beta(a,b,t)}{\beta(a,b,c)}$ and $G(t) = F(t)F(y) - F(\frac{ty}{c})$ for all $t > 0$.

Then, for all $t > 0$,

$$\begin{aligned} G'(t) &= F'(t)F(y) - \frac{y}{c}F'(\frac{ty}{c}) \\ &= \frac{g(t)F(y)}{\beta(a,b,c)}\left(\frac{yg(\frac{ty}{c})}{cF(y)g(t)} - 1\right) \\ &= \frac{g(t)F(y)}{\beta(a,b,c)}\left(\frac{y^a}{c^aF(y)}\left(\frac{1+t}{1+\frac{ty}{c}}\right)^{a+b} - 1\right). \end{aligned}$$

We note that $\left(\dfrac{1+t}{1+\frac{ty}{c}}\right)^{a+b}$ is decreasing in $t > 0$ since $y > c$.

Let $H(t) = \dfrac{y^a}{c^aF(y)}\left(\dfrac{1+t}{1+\frac{ty}{c}}\right)^{a+b} - 1$ for all $t > 0$. Then H is decreasing.

We note that $G(c) = F(c)F(y) - F(y) = 0$ and $\lim_{t\to\infty} G(t) = 0$.

By Roll's theorem, there is a point $p \in (c, \infty)$ such that $G'(p) = 0$. Then $H(p) = 0$. Then $H(t) > 0$ for all $t \in (c, p)$ and $H(t) < 0$ for all $t \in (p, \infty)$. Then $G'(t) > 0$ for all $t \in (c, p)$ and $G'(t) < 0$ for all $t \in (p, \infty)$. This implies that $G(x) \geq 0$. Then $F(x)F(y) \geq F(\frac{xy}{c})$. Hence, we obtain the inequality (4). □

We note on Theorem 2.1 that if $c = 1$ then we obtain the inequality (1).

Theorem 2.2. *Let $a, b > 0$ and $0 < y < c < x$. Then*

$$\beta(a,b,x)\beta(a,b,y) \le \beta(a,b,\frac{xy}{c})\beta(a,b,c). \tag{5}$$

Proof. Let $g(t) = \dfrac{t^{a-1}}{(1+t)^{a+b}}$, $F(t) = \dfrac{\beta(a,b,t)}{\beta(a,b,c)}$ and $G(t) = F(\frac{ty}{c}) - F(t)F(y)$ for all $t > 0$.

Then, for all $t > 0$,

$$\begin{aligned} G'(t) &= \frac{y}{c}F'(\frac{ty}{c}) - F'(t)F(y) \\ &= \frac{g(t)F(y)}{\beta(a,b,c)}\left(1 - \frac{yg(\frac{ty}{c})}{cF(y)g(t)}\right) \\ &= \frac{g(t)F(y)}{\beta(a,b,c)}\left(1 - \frac{y^a}{c^aF(y)}\left(\frac{1+t}{1+\frac{ty}{c}}\right)^{a+b}\right). \end{aligned}$$

We note that $\left(\dfrac{1+t}{1+\frac{ty}{c}}\right)^{a+b}$ is increasing in $t > 0$ since $y < c$.

Let $H(t) = 1 - \dfrac{y^a}{c^aF(y)}\left(\dfrac{1+t}{1+\frac{ty}{c}}\right)^{a+b}$ for all $t > 0$. Then H is decreasing.

We note that $G(c) = F(y) - F(c)F(y) = 0$ and $\lim_{t\to\infty} G(t) = 0$.

By Roll's theorem, there is a point $p \in (c, \infty)$ such that $G'(p) = 0$. Then $H(p) = 0$. Then $H(t) > 0$ for all $t \in (c, p)$ and $H(t) < 0$ for all $t \in (p, \infty)$. Then $G'(t) > 0$ for all $t \in (c, p)$ and $G'(t) < 0$ for all $t \in (p, \infty)$. This implies that $G(x) \ge 0$. Then $F(\frac{xy}{c}) \ge F(x)F(y)$. Hence, we obtain the inequality (5). □

We note on Theorem 2.2 that if $c = 1$ then we obtain the inequality (2).

Theorem 2.3. *Let $0 < a < 1$, $b > 0$, $0 \le c < y$ and $x > c$. Then*

$$\beta(a,b,x)\beta(a,b,y) \le \beta(a,b,x+y-c)\beta(a,b,c). \tag{6}$$

Proof. For any $t > 0$, we let $g(t) = \dfrac{t^{a-1}}{(1+t)^{a+b}}$, $F(t) = \dfrac{\beta(a,b,t)}{\beta(a,b,c)}$ and $G(t) = F(t+y-c) - F(t)F(y)$.

Then, for all $t > 0$,

$$\begin{aligned} G'(t) &= F'(t + y - c) - F'(t)F(y) \\ &= \frac{g(t)F(y)}{\beta(a,b,c)} \left(1 - \frac{g(t + y - c)}{F(y)g(t)}\right) \\ &= \frac{g(t)F(y)}{\beta(a,b,c)} \left(1 - \frac{1}{F(y)} \left(1 + \frac{y-c}{t}\right)^{a-1} \left(1 + \frac{y-c}{1+t}\right)^{-a-b}\right). \end{aligned}$$

We note that $\left(1 + \frac{y-c}{t}\right)^{a-1} \left(1 + \frac{y-c}{1+t}\right)^{-a-b}$ is increasing in $t > 0$ since $a < 1$ and $y > c$.

Let $H(t) = 1 - \frac{1}{F(y)} \left(1 + \frac{y-c}{t}\right)^{a-1} \left(1 + \frac{y-c}{1+t}\right)^{-a-b}$ for all $t > 0$. Then H is decreasing.

We note that $G(c) = F(y) - F(c)F(y) = 0$ and $\lim_{t\to\infty} G(t) = 0$.

By Roll's theorem, there is a point $p \in (c, \infty)$ such that $G'(p) = 0$. Then $H(p) = 0$. Then $H(t) > 0$ for all $t \in (c, p)$ and $H(t) < 0$ for all $t \in (p, \infty)$. Then $G'(t) > 0$ for all $t \in (c, p)$ and $G'(t) < 0$ for all $t \in (p, \infty)$. This implies that $G(x) \geq 0$. Then $F(x+y-c) \geq F(x)F(y)$. Hence, we obtain the inequality (6). □

We note on Theorem 2.3 that if $c = 0$ then we obtain the inequality (3).

References

[1] W. T. Sulaiman, Functional inequalites for incomplete beta and gamma functions, J. Inequal. Spec. Func., 2010, **1**(1), 10–15.

Received: February, 2013