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Three inequalities for the digamma function

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Abstract

In this paper, we prersent three inequalities involving the digamma function.

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1 Introduction

The gamma function Γ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,$$

where $x > 0$.

The digamma function Ψ is defined by

$$\Psi(x) = \frac{d}{dx}\ln\Gamma(x),$$

where $x > 0$.

Abramowitz and Stegun [1] showed that, for all $x > 0$,

$$\Psi(x) = -\gamma + \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{x+k}\right),$$

where γ is the Euler constant. Then, for all $x > 0$,

$$\Psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}}.$$

where n is a positive integer.

In 2011, Sulaiman [2] gave the inequalities as follows.

$$\Psi(x+y) \geq \Psi(x) + \Psi(y) \tag{1}$$

where $x > 0$ and $0 < y < 1$.

$$\Psi^{(n)}(x+y) \leq \Psi^{(n)}(x) + \Psi^{(n)}(y) \tag{2}$$

where n is a positive odd integer and $x, y > 0$.

$$\Psi^{(n)}(x+y) \geq \Psi^{(n)}(x) + \Psi^{(n)}(y) \tag{3}$$

where n is a positive even integer and $x, y > 0$.

In this paper, we present the generalizations for the inequalities (1), (2) and (3).

2 Results

Theorem 2.1. *Assume that* $x > 0$ *and* $0 < y_i \leq 1$ *for all* $i \in \mathbb{N}_m$. *Then*

$$\Psi(x + \sum_{i=1}^{m} y_i) \geq \Psi(x) + \sum_{i=1}^{m} \Psi(y_i). \tag{4}$$

Proof. Let $f(x) = \Psi(x + \sum_{i=1}^{m} y_i) - \Psi(x) - \sum_{i=1}^{m} \Psi(y_i)$. Then

$$f'(x) = \Psi'(x + \sum_{i=1}^{m} y_i) - \Psi'(x) = \sum_{k=0}^{\infty} \left(\frac{1}{(x + \sum_{i=1}^{m} y_i + k)^2} - \frac{1}{(x+k)^2} \right) \leq 0.$$

Hence, f is non-increasing. Moreover,

$$\lim_{x\to\infty} f(x) = m\gamma + \lim_{x\to\infty}\sum_{k=0}^{\infty}\left(\frac{-m}{k+1} - \frac{1}{x+\sum_{i=1}^{m} y_i + k} + \frac{1}{x+k} + \sum_{i=1}^{m}\frac{1}{y_i+k}\right)$$
$$= m\gamma + \sum_{k=0}^{\infty}\left(\frac{-m}{k+1} + \sum_{i=1}^{m}\frac{1}{y_i+k}\right)$$
$$= m\gamma + \sum_{k=0}^{\infty}\sum_{i=1}^{m}\frac{1-y_i}{(k+1)(y_i+k)} \geq 0.$$

This implies that $f(x) \geq 0$. Hence, we obtain the inequality (4). □

Theorem 2.2. *Let n be a positive odd integer. Assume that $x > 0$ and $y_i > 0$ for all $i \in \mathbb{N}_m$. Then*

$$\Psi^{(n)}\left(x + \sum_{i=1}^{m} y_i\right) \leq \Psi^{(n)}(x) + \sum_{i=1}^{m}\Psi^{(n)}(y_i). \tag{5}$$

Proof. Let $f(x) = \Psi^{(n)}(x) + \sum_{i=1}^{m}\Psi^{(n)}(y_i) - \Psi^{(n)}\left(x + \sum_{i=1}^{m} y_i\right)$. Then

$$f'(x) = \Psi^{(n+1)}(x) - \Psi^{(n+1)}\left(x + \sum_{i=1}^{m} y_i\right)$$
$$= (n+1)!\sum_{k=0}^{\infty}\left(-\frac{1}{(x+k)^{n+2}} + \frac{1}{(x+\sum_{i=1}^{m} y_i + k)^{n+2}}\right) \leq 0.$$

Hence, f is non-increasing. Moreover,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} n!\sum_{k=0}^{\infty}\left(\frac{1}{(x+k)^{n+1}} + \sum_{i=1}^{m}\frac{1}{(y_i+k)^{n+1}} - \frac{1}{(x+\sum_{i=1}^{m} y_i + k)^{n+1}}\right)$$
$$= n!\sum_{k=0}^{\infty}\sum_{i=1}^{m}\frac{1}{(y_i+k)^{n+1}} \geq 0.$$

This implies that $f(x) \geq 0$. Hence, we obtain the inequality (5). $\square$

Theorem 2.3. *Let n be a positive even integer. Assume that $x > 0$ and $y_i > 0$ for all $i \in \mathbb{N}_m$. Then*

$$\Psi^{(n)}\left(x + \sum_{i=1}^{m} y_i\right) \geq \Psi^{(n)}(x) + \sum_{i=1}^{m} \Psi^{(n)}(y_i). \tag{6}$$

Proof. Let $f(x) = \Psi^{(n)}\left(x + \sum_{i=1}^{m} y_i\right) - \Psi^{(n)}(x) - \sum_{i=1}^{m} \Psi^{(n)}(y_i)$. Then

$$\begin{aligned} f'(x) &= \Psi^{(n+1)}\left(x + \sum_{i=1}^{m} y_i\right) - \Psi^{(n+1)}(x) \\ &= (n+1)! \sum_{k=0}^{\infty} \left(\frac{1}{(x + \sum_{i=1}^{m} y_i + k)^{n+2}} - \frac{1}{(x+k)^{n+2}} \right) \leq 0. \end{aligned}$$

Hence, f is non-increasing. Moreover,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} n! \sum_{k=0}^{\infty} \left(-\frac{1}{(x + \sum_{i=1}^{m} y_i + k)^{n+1}} + \frac{1}{(x+k)^{n+1}} + \sum_{i=1}^{m} \frac{1}{(y_i+k)^{n+1}} \right)$$

$$= n! \sum_{k=0}^{\infty} \sum_{i=1}^{m} \frac{1}{(y_i+k)^{n+1}} \geq 0.$$

This implies that $f(x) \geq 0$. Hence, we obtain the inequality (6). $\square$

References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas Graphs and Mathematical Tables, New York, 1972, 258–259.

[2] W. T. Sulaiman, Turan inequalites for the digamma and polygamma functions, South Asian J. Math., 2011, **1**(2), 49–55.

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