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A Unique Common Fixed Point of a pair of Self-maps on a 2-metric space

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Abstract

Let (M, ρ) be a complete metric space and f a self-map on M such that $\rho(fx, fy) \leq \beta\rho(fx, fy)$ for all $x, y \in X$, where $0 \leq \beta < 1/2$. Kannan proved that f has a unique fixed point p and for each $x \in M$ the iterates $f, f^2, \ldots$ will converge to p. In this paper, we extend this result to a pair of self-maps on a complete 2-metric space. Our technique is remarkable to use only elementary properties of greatest lower bound, and repeatedly employing the *symmetry* and the *tetrahedron inequality* of the 2-metric instead of routine iteration procedure. This idea was initiated for only metric spaces

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1. Introduction

Let (M, ρ) be a metric space. According to the well-known Banach contraction mapping theorem, a contraction f on M which enjoys the property that $\rho(fx, fy) \leq \alpha d(x, y)$ for all $x, y \in M$ where $0 < \alpha < 1$, will have a unique fixed point, provided M is complete. Banach proved that the *f-orbital sequence* $fx, f^2x, \ldots$ at each $x \in M$ is a Cauchy in M and hence converges to some $p \in M$. Therefore by the continuity of the contraction f, $p = \lim_{n\to\infty} f^{n+1}x = f(\lim_{n\to\infty} f^n x) = fp$. In other words, p is a contractive fixed point [14]. Several generalizations of contraction theorem were established by either weakening the contraction condition and/or relaxing the completeness of M (For example, see [2]-[5] and [9]). Kannan [11] obtained a unique fixed point without the continuity of f by replacing the contraction condition with $d(fx, fy) \leq \beta[d(x, fx) + d(y, fy)]$ for all $x, y \in M$, where $0 \leq \beta < 1/2$. Interestingly, Kannan's theorem is independent of the Banach contraction principle. Also, Kannan's fixed point theorem is very important because it characterizes the metric completeness [16]. That is, a metric space M is complete if and only if every Kannan mapping on M has a fixed point. *Iteration technique is a fundamental characteristic in all these results*.

The purpose of this paper is to prove Kannan's result [11] for two self-maps in the setting of 2-metric spaces, using only some elementary properties of greatest lower bound. It is highly significant to see that the traditional iteration technique is not used.

2. Preliminaries

This section devotes to some of the elementary concepts and prerequisites.

Definition 2.1 Let X be a non-empty set and d, a real valued function defined on $X^3 = X \times X \times X$ satisfying the following axioms:

(a) Given a pair of distinct elements $x, y \in X$, there exists a $z \in X$ with $d(x, y, z) > 0$

(b) $d(x, y, z) = 0$ whenever at least two of x, y and z are equal in X

(c) $d(x,y,z) = d(x,z,y) = d(y,x,z) = d(z,x,y) = d(y,z,x) = d(z,y,x)$ for all $x, y, z \in X$

(d) $d(x,y,z) \le d(x,y,w) + d(x,w,z) + d(w,y,z)$ for all $x, y, z, w \in X$.

Then d is called a 2-metric on X and the pair (X, d) a 2-*metric space* [6].

It is obvious from the axioms (a) and (b) that a 2-metric d is non-negative.

It can be easily seen that

Remark 2.1 If $(x, y) \in X$ is such that $d(x, y, z) = 0$ for all $z \in X$, then $x = y$.

Axiom (c) means that the value of $d(x, y, z)$ is independent of the order of x, y and z, and is usually known as the *symmetry* of d under a permutation on x, y and z.

Example 2.1 Consider $X = \mathbb{R}^2$ with metric $\rho(x, y) = \|x - y\|$ for all $x, y, \in X$. Define

$$d(x,y,z) = \tfrac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \quad \text{for all } x, y, z \in X$$

where $a = \rho(x, y)$, $b = \rho(y, z)$ and $c = \rho(z, x)$.

Geometrically d represents the area of a triangle with vertices x, y and z in the plane and is usually is referred to as *area* metric.

In view of the fact that the area of a triangle face of a tetrahedron does not exceed the sum of the areas of the remaining faces, Axiom (d) is referred to as the *tetrahedron inequality*.

The research study on 2-metric space initiated by Gahler [6, 7] has been extended by many others through various types of contraction and non-expansive type conditions (See for instance, [8], [10], [12], [13] and [15]).

Definition 2.2 A sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is said to be 2-*convergent* with limit $p \in X$ if $\lim_{n\to\infty} d(x_n, p, z) = 0$ for all $z \in X$ and is denoted by $x_n \to p$ as $n \to \infty$.

Using the tetrahedron inequality (d) and Definition 2.2, it is hard to prove that a 2-metric d is *sequentially* continuous in any one of the coordinate variables x, y and z.

Remark 2.2 A 2-metric may be continuous in any one of variables x, y and z without being continuous in all the three variables (*cf.* Example 0.2, [13]).

Definition 2.3 A 2-metric d is said to be *continuous* if it is continuous in two of and hence all the three variables.

Definition 2.4 A sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is said to 2-*Cauchy* if $\lim_{n\to\infty} d(x_n, x_{n+k}, z) = 0$ for all integers $k \geq 1$ for all $z \in X$.

Remark 2.3 A 2-convergent sequence need not be 2-Cauchy (*cf.* Ex. 0.2, [13]).

Definition 2.5 A 2-metric space X is said to be *complete* if every 2-Cauchy sequence in it is 2-convergent with limit in it [13].
Remarks 2.1, 2.2 and 2.3 reveal that these two notions of metric and 2-metric seem to be unrelated.

3. Main Results

Let S be a non empty set of non negative real numbers which is bounded below. Then by the infimum property of R (*cf.* § 2.4, [1]), S will have a greatest lower bound, say a in R. We immediately see that any number in S which exceeds a cannot be a lower bound of S.

Here after X denotes a 2-metric space equipped with 2-metic d and f a self-map on X.

We now prove

Theorem 3.1 *Suppose that there exists a constant* β *such that* $0 \leq \beta < 1/2$ and

$$d(fx, gy, z) \leq \beta[d(x, fx, z) + d(y, gy, z)] \text{ for all } x, y, z \in X. \quad \ldots \quad (3.1)$$

If X is complete, then f and g will have a unique common fixed point.

Proof. Let $S_z(f) = \{d(x, fx, z) : x \in X\}$ and $S_z(g) = \{d(x, gx, z) : x \in X\}$

indexed by $z \in X$.

Clearly $S_z(f) \cup S_z(g)$ is a non empty set of non negative real numbers which is bounded below by 0. Hence it has a greatest lower bound, say a_z.

We discuss two exhaustive cases:

Case (a): $a_z = \text{glb}\, S_z(f)$.

If possible suppose that $a_z > 0$ for some $z \in X$. Writing $y = fx$ in (3.1),

$$d(fx,gfx,z)\le\beta[d(x,fx,z)+d(fx,gfx,z)]$$

or $$d(fx,gfx,z)\le\left(\frac{\beta}{1-\beta}\right)d(x,fx,z). \qquad \dots \quad (3.2)$$

The choice of β implies that $\frac{\beta}{1-\beta}<1$ so that from (3.2) it follows that $d(fx,gfx,z)<a_z$ for some $x\in X$, which contradicts with the choice of a_z.

Therefore $a_z=0$ for each $z\in X$. Hence we can find points $x_1,x_2,\dots,x_n,\dots\in X$ such that $d(x_n,fx_n,z)\in S_z(f)$ for n = 1, 2, … and

$$\lim_{n\to\infty}d(x_n,fx_n,z)=0 \text{ for each } z\in X. \qquad \dots \quad (3.3)$$

In fact, given $z\in X$ note that (3.3) is obvious if $d(x_n,fx_n,z)=0$ for all n. Therefore we assume that $d(x_{n_0}fx_{n_0},z)>0$ for some $n_0\in\mathbb{N}$.

If $d(x_{n_0}fx_{n_0},z)\le d(x_kfx_k,z)$ for all $k\ge n_0$, we get $d(x_{n_0}fx_{n_0},z)>0$ would become a lower bound of $S_z(f)$ contradicting that 0 is its greatest lower bound. Thus we can choose $n_1<n_0$ such that $d(x_{n_1}fx_{n_1},z)<d(x_{n_0}fx_{n_0},z)$. By induction we can choose $n_0>n_1>\cdots>n_k>\cdots$ such that $d(x_{n_0}fx_{n_0},z)>d(x_{n_1}fx_{n_1},z)>\cdots>d(x_{n_k}fx_{n_k},z)>\cdots$

Thus $\langle d(x_{n_k},fx_{n_k},z)\rangle_{k=1}^{\infty}$ being a monotonically decreasing sequence of non negative numbers in S_z converges to the infimum 0, proving (3.2) for each $z\in X$.

$$\lim_{n\to\infty}d(x_n,fx_n,x_{n+k})=0 \text{ for each positive integer } k\ge 1. \qquad \dots \quad (3.4)$$

Again by (3.1), the symmetry and the tetrahedron inequality of d, we see that

$$\begin{aligned} d(x_n,gx_n,z) &\le d(x_n,gx_n,fx_n)+d(x_n,fx_n,z)+d(fx_n,gx_n,z)\\ &= d(fx_n,gx_n,x_n,)+d(fx_n,gx_n,z)+d(x_n,fx_n,z)\\ &\le \beta[d(x_n,fx_n,x_n)+d(x_n,gx_n,x_n)]\\ &\qquad + \beta[d(x_n,fx_n,z)+d(x_n,gx_n,z)] + d(x_n,fx_n,z)\\ &= \beta[d(x_n,fx_n,z)+d(x_n,gx_n,z)] + d(x_n,fx_n,z)\end{aligned}$$

or $$d(x_n,gx_n,z)\le\left(\frac{1+\beta}{1-\beta}\right)d(x_n,fx_n,z).$$

Thus in view of (3.3), it follows that

$$\lim_{n\to\infty}d(x_n,gx_n,z)=0 \text{ for each } z\in X. \qquad \dots \quad (3.5)$$

In particular

$$\lim_{n\to\infty}d(x_n,gx_n,x_{n+k})=0 \text{ for each positive integer } k\ge 1. \qquad \dots \quad (3.6)$$

Next we establish that

(e) $\langle x_n\rangle_{n=1}^{\infty}\subset X$ is 2-Cauchy.

We use (2-m3) and (2-m4) repeatedly to get

$$\begin{aligned} d(x_n,x_{n+k},z) &\le d(x_n,x_{n+k},fx_n)+d(x_n,fx_n,z)+d(fx_n,x_{n+k},z) \\ &\le d(x_n,fx_n,x_{n+k})+d(x_n,fx_n,z) \\ &\qquad +[d(fx_n,x_{n+k},gx_{n+k})+d(fx_n,gx_{n+k},z)+d(gx_{n+k},x_{n+k},z)] \\ &= d(x_n,fx_n,x_{n+k})+d(x_n,fx_n,z)+d(fx_n,gx_{n+k},x_{n+k}) \\ &\qquad +d(fx_n,gx_{n+k},z)+d(x_{n+k},gx_{n+k},z) \\ &\le d(x_n,fx_n,x_{n+k})+d(x_n,fx_n,z)+\beta[d(x_n,fx_n,x_{n+k})+d(x_{n+k},gx_{n+k},x_{n+k})] \\ &\qquad +\beta[d(x_n,fx_n,z)+d(x_{n+k},gx_{n+k},z)]+d(x_{n+k},gx_{n+k},z) \\ &\le (1+\beta)[d(x_n,fx_n,x_{n+k})+d(x_n,fx_n,z)]+d(x_{n+k},gx_{n+k},z)] \\ &\qquad +\beta d(x_{n+k},gx_{n+k},z)\,. \end{aligned}$$

Applying limit as $n\to\infty$ in this, and then using (3.3)-(3.6), we obtain (e).

Since X is complete, $\langle x_n\rangle_{n=1}^{\infty}$ is 2-convergent with limit $p\in X$, that is

$$\lim_{n\to\infty} d(x_n,p,z)=0 \quad \text{for all } z\in X. \qquad \ldots \qquad (3.7)$$

To prove that p is a fixed point of f, again by (2-m3) and (2-m4) and (3.1),

$$\begin{aligned} d(fp,p,z) &\le d(fp,p,x_n)+d(fp,x_n,z)+d(x_n,p,z) \\ &= d(x_n,p,fp)+d(fp,x_n,z)+d(x_n,p,z) \\ &\le d(x_n,p,fp)+[d(fp,x_n,gx_n)+d(fp,gx_n,z)+d(gx_n,x_n,z)]+d(x_n,p,z) \\ &= d(x_n,p,fp)+d(fp,gx_n,x_n)+d(fp,gx_n,z)+d(x_n,gx_n,z)+d(x_n,p,z) \\ &\le d(x_n,p,fp)+\beta[d(p,fp,x_n)+d(x_n,gx_n,x_n)] \\ &\qquad +\beta[d(p,fp,z)+d(x_n,gx_n,z)]+d(x_n,gx_n,z)+d(x_n,p,z) \end{aligned}$$

or $(1-\beta)d(fp,p,z)\le(1+\beta)[d(x_n,p,fp)+d(x_n,gx_n,z)]+d(x_n,p,z)$.

As $n\to\infty$, this together with (3.3)-(3.7) yields $d(fp,p,z)\le 0$ for all $z\in X$ which in view of Remark 2.1 implies that $fp=p$, that is p is a fixed point of f.

Again writing $x=y=p$ in (3.1), we have

$$d(p,gp,z)=d(fp,gp,z)\le\beta[d(p,fp,z)+d(p,gp,z)]=\beta d(p,gp,z)$$

Or $(1-\beta)d(fp,gp,z)\le 0$ so that $d(fp,gp,z)=0$, since $\beta<1$. Thus by Remark 2.1, we find that p is a fixed point of g as well.

Case (b): $a_z=\mathrm{glb}S_z(g)$. This is entirely similar to Case (a).

Uniqueness: Let q be another common fixed point of f and g.
Then from (3.1), we get

$$d(p,q,z)=d(fp,gq,z)\le\beta[d(p,fp,z)+d(q,gq,z)]=0 \quad \text{for all } z\in X.$$

Again by Remark 2.1, it follows that $p=q$. That is the common fixed point of f and g is unique. □

Taking $g=f$ in Theorem 3.1, We now have

Theorem 3.2 *Suppose that there exists a constant* β *such that* $0 \le \beta < 1/2$ and

$$d(fx, fy, z) \le \beta[d(x, fx, z) + d(y, fy, z)] \text{ for all } x, y, z \in X. \qquad \ldots (3.8)$$

If X is complete, then f has a unique fixed point.

At this juncture, we introduce the following notion:

Definition 3.2 A fixed point p of f on X is a 2-*contractive fixed point* of it if the orbital sequence $x, fx, \ldots, f^n x, \ldots$ at each $x \in X$ converges to p.

We prove that the unique fixed point p will be a 2-*contractive fixed point* of it as given below.

Taking $y = p = fp$ in (3.1) and simplifying, it follows that

$$d(f^n x, p, z) = d(f^n x, g^n p, z) \le \beta[d(f^{n-1}x, f^n x, z) + d(g^{n-1}p, g^n p, z)]$$
$$= \beta d(f^{n-1}x, f^n x, z). \qquad \ldots (3.9)$$

But again by (3.1), we have

$$d(f^{n-1}x, f^n x, z) \le \beta[d(f^{n-2}x, f^{n-1}x, z) + d(f^n x, f^{n-1}x, z)]$$

or $$d(f^{n-1}x, f^n x, z) \le \left(\frac{\beta}{1-\beta}\right) d(f^{n-2}x, f^{n-1}x, z).$$

Hence by the induction we get

$$d(f^{n-1}x, f^n x, z) \le \left(\frac{\beta}{1-\beta}\right)^{n-1} d(x, fx, z) \text{ for all } x, z \in X \text{ and all } n \ge 1. \qquad \ldots (3.10)$$

Substituting (3.10) in (3.9), it follows that

$$d(f^n x, p, z) \le \beta\left(\frac{\beta}{1-\beta}\right)^{n} d(x, fx, z) \text{ for all } x, z \in X \text{ and all } n \ge 1. \qquad \ldots (3.11)$$

Since $\beta < 1/2$, we note that $\frac{\beta}{1-\beta} < 1$ so that $\left(\frac{\beta}{1-\beta}\right)^n \to 0 \quad n \to \infty$.

Therefore applying the limit as $n \to \infty$ in (3.8) and using this finally we get $d(f^n x, p, z) \to 0$ as $n \to \infty$ for all $x, z \in X$.
In other words, p is a 2-contractive fixed point of f. □

We see that Theorem 3.1 is an analogue of the following result proved by Kannan [11] for metric spaces:

Theorem 3.3 *Let f be a self-map on a complete metric space* (M, ρ) *satisfying the condition* $d(fx, fy) \le \beta[d(x, fx) + d(y, fy)]$ *for all* $x, y \in X$*, where* $0 < \beta < 1$. *Then g will have a unique fixed point to which all the g-orbits converge.*

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