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Rheological Response and Validity of Viscoelastic Model through Wave Propagation

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Abstract

The effect of non-homogeneity on wave propagation in four parameter viscoelastic model is investigated analytically as well as numerically. The non-homogeneity parameters in viscoelastic rods are taken as dependent of space coordinates. The consecutive equation of four parameter viscoelastic model is developed and then it is solved with the help of Friedlander series and Eikonal equation of optics. The asymptotic equation of wave front for harmonic waves in non-homogeneous viscoelastic rods are obtained by reducing the linear partial differential equation into ordinary differential equation. The problem is illustrated graphically in detail.

Mathematics Subject Classification: 74C10; 74C15; 74C20; 74E05; 74E20

Keywords: harmonic waves; friedlander series; four parameter model.

1. Introduction

Modeling problems actually is a study of mechanical properties of materials. The polymer models have specific characteristics which distinguish them from elastic models. The elastic materials store maximum of the energy (100%) due to deformation but viscoelastic materials do not do this. The dissipation of energy in polymer materials is known as hysteresis. Nearly, all materials behave like some viscoelastic response. However, some common materials such as steel or quartz do not deviate much from linear elasticity at room temperature. But synthetic rubber, wood, and biological tissue and metals at high temperature show significant viscoelastic effects [1].

The theory of elasticity is formulated and developed by Alfrey [2], Barberan [3], Achenbach [4], Bhattacharya [5] and Acharya [6]. Further, Bert [7], Biot [8], Batra [9] successfully applied this theory to wave-propagation in homogeneous, elastic media. On the basis of the theory of elasticity, the propagation of harmonic waves in isotropic or anisotropic materials has been evaluated numerically by White [10], Mirsky [11] and Tsai [12]. To explain the soil behaviour, Murayama [13] and Schiffman et al. [14] developed five and seven parameter models. Moodie [15] presented research paper on propagation, reflection and transmission of transient cylindrical shear waves in non-homogeneous four-parameter viscoelastic media.

The authors have studied four and five parameter viscoelastic models for wave propagation and dynamic loading [16-21]. But in this study, we consider, the specimen is non-homogeneous i.e. density, rigidity and viscosity of the rod is space dependent In this paper, the wave equation is approximated by using WKB theory. The displacements are assumed to be small under isothermal conditions, the linear constitutive laws hold. Time dependent displacement and stress boundary conditions are employed for calculating the relations for displacement and stress. The rods are assumed to be initially unstressed and at rest. In this study, it is assumed that density $'\rho'$, rigidity $'G'$ and viscosity $'\eta'$ of the specimen i.e. rod are space dependent and obey the harmonic laws as $\rho=\rho_0 e^{2i\alpha_1 x}$, $k=k_0 e^{2i\alpha_2 x}$, $\eta=\eta_0 e^{2i\alpha_3 x}$. The various graphs are plotted to show the effect of non-homogeneity on the velocity of waves.

2. Formulation of Problem

Let us consider wave is propagating in one dimensional non-homogeneous semi-infinite rod, the end of the rod is kept at x=0. We consider the four parameter model with two springs $S_1(G_1), S_2(G_2)$ and two dash-pots $D_1(\eta_1), D_2(\eta_2)$ with viscoelasticity η_1 and η_2 respectively (Fig.1). The springs represent recoverable elastic response and dash pot represents elements in the structure giving rise to viscous drag. Here G_1 and G_2 are elastic parameters, η_1 and η_2 are viscoelastic parameters. Let σ be the stress and a be the strain in the model. Let a_1 be the strain in $S_1(G_1)$, a_2 be the strain along dashpot $D_1(\eta_1)$ and a_3 be the strain in the Kelvin model. Fig. 1 represents the sketch of the standard four parameter viscoelastic models. The stress v/s strain behavior for constant stress (σ) with time (t_a) has been shown in fig. 1. Here, $G_1=\lambda_1+2\mu_1$, $G_2=\lambda_2+2\mu_2$ are the modulli of elasticity, η_1, η_2 are Newtonian viscosities coefficients and taken as functions of 'x' in the non-homogeneous case.

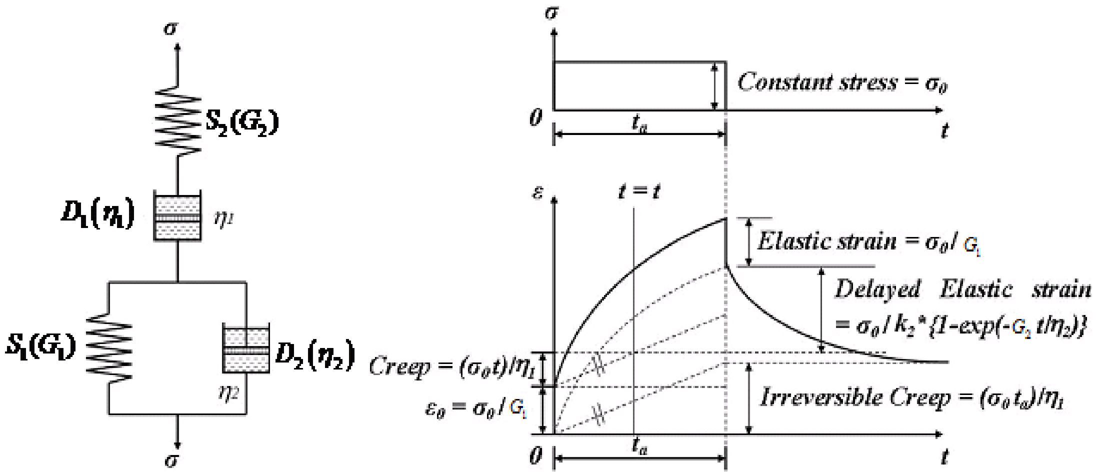


Fig. 1 Rheological model and its response

The stress-strain relation for the four parameter viscoelastic models are constituted by the equations [18]

$$\sigma = G_1 a_1, \sigma = \eta_1 \dot{a}_2, \sigma = G_2 a_3, \sigma = \eta_2 \dot{a}_3. \tag{1}$$

Eliminating a_1, a_2, a_3 from Eq. (1) we get the constitute equation for the four parameter model:

$$\ddot{\sigma} + \left(\frac{G_1}{\eta_2} + \frac{G_2}{\eta_2} + \frac{G_1}{\eta_1} \right) \dot{\sigma} + \left(\frac{G_1 G_2}{\eta_1 \eta_2} \right) \sigma = G_1 \ddot{a} + \frac{G_1 G_2}{\eta_2} \dot{a} \tag{2}$$

The stress strain equation for four parameter model is of general form:

$$\ddot{\sigma} + B_1 \dot{\sigma} + B_0 \sigma = A_2 \ddot{a} + A_1 \dot{a} \tag{3}$$

Where B_i's and A_i's are the coefficients made up of combinations of the G_1, G_2 and η_1, η_2 and depend upon the specific arrangement of the elements in the model. In operator form the Eq. (2) can be written as

$$\left\{ \partial_t^2 + B_1 \partial_t + B_0 \right\} \sigma = \left\{ A_2 \partial_t^2 + A_1 \partial_t \right\} a \tag{4}$$

The equation of motion and strain-displacement relation is given by

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 U}{\partial t^2} \tag{5}$$

$$a = \frac{\partial U}{\partial x} \tag{6}$$

Where $\rho = \rho(x)$ is the variable density of the material.

Differentiating Eq.(5) w.r.t x ,we get

$$-\frac{1}{\rho^2} \frac{\partial \rho}{\partial x} \frac{\partial \sigma}{\partial x} + \frac{1}{\rho} \frac{\partial^2 \sigma}{\partial x^2} = u_{,xtt} \tag{7}$$

Differentiating Eq.(6) w.r.t. t ,we get

$$\frac{\partial a}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)$$

Again differentiating w.r.t. t,

$$\frac{\partial^2 a}{\partial t^2}=\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\right\}=u_{,ttx}=u_{,xtt} \tag{8}$$

Using Eq. (7) and Eq. (8), Eq. (3) gives

$$\sigma_{,ttt}+,\ B_1\sigma_{,tt}+B_0\sigma_{,t}=\frac{1}{\rho}\left\{A_2\sigma_{,xxt}-A_2\left(log\rho\right)_{,x}\sigma_{,xt}+A_1\sigma_{,xx}-A_1(log\rho)_{,x}\sigma_{,x}\right\} \tag{9}$$

3. Method of Solution

Let the solution $\sigma(x,t)$ of Eq. (9) may be represented by the series [14]

$$\sigma\left(x,t\right)=\sum_{n=0}^{\infty}A_n\left(x\right)F_n\left\{t-h\left(x\right)\right\},\qquad A_0\neq 0 \tag{10}$$

Where,

$F'_n=F_{n-1}$ (where, n = 1, 2, 3…………) with $F_{n,t}=F_{n-1}$ and $F_{n,x}=-h_{,x}F_{n-1}$ (11)

and for $n<0$ assume that $A_n=0$ and the derivatives of σ may be obtained by term-wise differentiation of Eq. (10), the prime in Eq. (11) denotes differentiation with respect to the argument concerned, and by using Eq. (10) and Eq.(11) we relate all $F_n^{'}$s to F_0 by successive integrations.

The Solution of equation Eq. (9) in the form of Eq. (10) can be obtained by taking a phase function $h(x)$, $h(x)$ satisfies the Eikonal equation of geometrical optics [15]

$$\left(\frac{dh(x)}{dx}\right)^2=\frac{\rho}{G_1}=\frac{1}{c^2} \tag{12}$$

Where c = c(x) is the variable wave speed for elastic longitudinal waves in a medium whose modulus of elasticity G_1 .Using, Eq.(10), Eq.(11) and the successive derivatives of $\sigma\left(x,t\right)$ *w.r.t.* *'t'* and *'x'* in equation Eq.(9), we get

$$\begin{aligned}&A_nF_{n-3}+\left(\frac{G_1}{\eta_2}+\frac{G_2}{\eta_2}+\frac{G_1}{\eta_1}\right)A_nF_{n-2}+\left(\frac{G_1G_2}{\eta_1\eta_2}\right)A_nF_{n-1}\\&=\frac{G_1}{\rho}\left\{A_n^{''}F_{n-1}-2A_n^{'}F_{n-2}h^{'}+A_nF_{n-3}\left(h^{'}\right)^2-A_nF_{n-2}h^{''}\right\}-\frac{G_1\rho^{'}}{\rho^2}\left(A_n^{'}F_{n-1}-A_nF_{n-2}h^{'}\right)\\&+\frac{G_1G_2}{\eta_2\rho}\left\{A_n^{''}F_n-2A_n^{'}F_{n-1}h^{'}+A_nF_{n-2}\left(h^{'}\right)^2-A_nF_{n-1}h^{''}\right\}-\frac{G_1G_2}{\eta_2}\frac{\rho'}{\rho^2}\left(A_n^{'}F_n-A_nF_{n-1}h^{'}\right)\end{aligned} \tag{13}$$

On simplifying the Eq. (13) using Eq. (11), we get the amplitude function satisfy the equation

$$2h^{'}(x)A^{'}_{n}(x)+\left\{\rho\left(\frac{1}{\eta_1}+\frac{1}{\eta_2}\right)-(log\rho)_{,x}h^{'}(x)+h''(x)\right\}A_n(x)=Q_n,(n=0,1,2\ldots) \tag{14}$$

Where,

$$Q_n=A''_{n-1}-\left\{\frac{1}{(h')^2}(log\rho)_{,x}+2\frac{G_2}{\eta_2}\right\}A'_{n-1}+\frac{G_2}{\eta_2}\left\{(h')^2(log\rho)_{,x}-\frac{G_1}{\eta_1}(h')^2-2h''\right\}A_{n-1}$$

$$+\frac{G_2}{\eta_2}A''_{n-2}-\frac{G_2}{\eta_2}(log\rho)_{,x}A'_{n-2}$$

Which is a linear partial differential equation and its solution is obtained by reducing it into ordinary differential equation using an asymptotic method. The origin of this method is the ray optics and central feature of this method is the motion of rays which are curves or straight lines. The rays are of the fundamental importance because all the functions which make up the various terms of the asymptotic expansion can be shown to satisfy ordinary differential equations along these curves. Thus, this is the one of the method which reduces partial differential equation to ordinary differential equation. Also on using asymptotic method, it is very important to choose the proper signs in the solution of equation (12) so that the direction of the propagation of the wave is taken into consideration.

On integrating Eq. (12), we get

$$h(x)=h(0)\pm\int_0^x\frac{ds}{c(s)} \tag{15}$$

The plus sign shows the wave travelling along positive direction of x-axis and negative sign shows the waves travelling along negative direction of x-axis. Therefore the solution of Eq. (14) can be obtained as

$$A^{'}_{n}(x)+\frac{1}{2}\left\{\rho c\left(\frac{1}{\eta_1}+\frac{1}{\eta_2}\right)-(log\rho)_{,x}+\frac{h''(x)}{h^{'}(x)}\right\}A_n(x)=\frac{c}{2}Q_n,(n=0,1,2\ldots)$$

$$A^{'}_{n}(x)+\frac{1}{2}\left\{\rho c\left(\frac{1}{\eta_1}+\frac{1}{\eta_2}\right)-(log\rho)_{,x}+(\log h,x),x\right\}A_n(x)=\frac{c}{2}Q_n,(n=0,1,2\ldots)$$

$$A^{'}_{n}(x)+\frac{1}{2}\left\{\rho c\left(\frac{1}{\eta_1}+\frac{1}{\eta_2}\right)-(log\rho c)_{,x}\right\}A_n(x)=\frac{c}{2}Q_n,(n=0,1,2\ldots)$$

$$A^{'}_{n}(x)+\left\{m(x)-\frac{1}{2}(\log l(s))_{,x}\right\}A_n(x)=\frac{c}{2}Q_n,(n=0,1,2\ldots)$$

Integrating Factor

$$e^{\int_0^x \left\{m(x)-\frac{1}{2}(\log l(s))_{,x}\right\}ds} = \log\left[\frac{l(0)}{l(x)}\right]^{1/2} \exp\left(\int_0^x m(s)ds\right)$$

And its solution is

$$A_n(x)\log\left[\frac{l(0)}{l(x)}\right]^{1/2} \exp\left(\int_0^x m(s)ds\right) = \frac{1}{2}\int_0^x c(s)Q_n(s).\log\left[\frac{l(0)}{l(x)}\right]^{1/2} \exp\left(\int_0^x m(s)ds\right)ds + r$$

Where r is constant of integration.

At $x=0, r=A(0)$

$$A_n(x) = A_n(0)\left\{\frac{l(x)}{l(0)}\right\}^{\frac{1}{2}} exp\left\{-\int_0^x m(s)ds\right\} + \frac{1}{2}\left\{\frac{l(x)}{l(0)}\right\}^{\frac{1}{2}} exp\left\{-\int_0^x m(s)ds\right\}\int_0^x c(s)\left\{\frac{l(0)}{l(s)}\right\}^{\frac{1}{2}} exp\left\{+\int_x^z m(z)dz\right\} Q_n^{\pm}(s)ds \tag{16}$$

Which is the expression for wave travelling in the positive direction of x-axis. The expression for wave travelling in positive and negative direction of x-axis is as

$$A_n(x) = A_n(0)\left\{\frac{l(x)}{l(0)}\right\}^{\frac{1}{2}} exp\left\{\mp\int_0^x m(s)ds\right\} \pm \frac{1}{2}\int_0^x c(s)\left\{\frac{l(x)}{l(s)}\right\}^{\frac{1}{2}} exp\left\{\pm\int_x^z m(z)dz\right\} Q_n^{\pm}(s)ds$$

$(n=0,1,2\ldots)$

Where, $l(x)=\rho c$ and $m(x)=\frac{\rho c}{2}\left(\frac{1}{\eta_1}+\frac{1}{\eta_2}\right).$ (17)

The upper signs are associated with wave traveling in the positive direction of x and the lower signs are associated with the waves travelling in the negative direction of x. At the end $x=0$,the impulse of magnitude σ_0 is suddenly applied and thereafter steadily maintained , that is

$$\sigma(0,t)=\sigma_0 H(t) \tag{18}$$

From Eq. (11) and Eq. (18), we have

$$\sum_{n=0}^{\infty} A_n(0)F_n\{t-h(0)\} = \sigma_0 H(t) \tag{19}$$

Thus we choose [15]

$$A_n(0) = \begin{vmatrix} \sigma_0 \cdots\cdots\cdots if\ n=0 \\ 0\cdots\cdots\cdots if\ n<0\, or\, n>0 \end{vmatrix} \tag{20}$$

$$h(0)=0 \text{ and } F_0 = H(t) \tag{21}$$

The solution of Eq. (9), for the waves travelling in the positive direction of x is generated by boundary stress Eq. (20), is

$$\sigma(x,t)=\sum_{n=0}^{\infty}A_n(x)\frac{\{t-h(x)\}^n}{n!}H\{t-h(x)\}=\sum_{n=0}^{\infty}A_n(x)\frac{\{t-h(x)\}^n}{n!}H\left\{t-\int_0^x\frac{ds}{c(s)}\right\} \quad (22)$$

Where,

$$h(x)=\int_0^x\frac{ds}{c(s)} \quad (23)$$

Where, $A_n(x)$ are given recursively by Eq. (16) (with upper signs) in combination with Eq. (20).

The first–term approximation leads to Eq. (23) as

$$\sigma(x,t)=\sigma_0\left\{\frac{l(x)}{l(0)}\right\}^{\frac{1}{2}}exp\left\{-\int_0^x m(s)ds\right\}H\left\{t-\int_0^x\frac{ds}{c(s)}\right\} \quad (24)$$

The Eq. (24) represents a transient stress wave which starts from the end '$x=0$' with amplitude 'σ_0' and moves in the positive direction of 'x' with velocity c(x). Hence, it is modulated by the factor

$$\left\{\frac{l(x)}{l(0)}\right\}^{\frac{1}{2}}exp\left\{-\int_0^x m(s)ds\right\} \quad (25)$$

Further terms in the approximate solution may be obtained recursively from Eq. (19)

4. Viscoelastic Model Applied to a Particular Case

For the sake of concreteness and for studying the qualitative effect of non-homogeneity on the longitudinal wave propagation in non-homogeneous four parameter viscoelastic rods, it is assumed that density 'ρ', rigidity 'G' and viscosity 'η' of the specimen i.e. rod are space dependent and obey the harmonic laws

$$\rho=\rho_0 e^{2i\alpha_1 x}\quad ,G=G_0 e^{2i\alpha_2 x},\eta=\eta_0 e^{2i\alpha_3 x} \quad (26)$$

If, $\alpha_1\ge\alpha_2\ge\alpha_3$ i.e. density $\ge$ rigidity $\ge$ viscosity (27)

Case-1

When, $\alpha_1=\alpha_2=\alpha_3$, then from Eq. (26), we get

$$\rho=\rho_0 e^{2i\alpha x}\quad ,G=G_0 e^{2i\alpha x},\eta=\eta_0 e^{2i\alpha x} \quad (28)$$

Therefore, from Eikonal equation of geometric optics

$$\left(\frac{dh(x)}{dx}\right)^2 = \frac{\rho}{G_1} = \frac{\rho_0 e^{2i\alpha x}}{G_{10} e^{2i\alpha x}} = \frac{\rho_0}{G_{10}} = \frac{1}{c_0{}^2} \quad = \text{constant.} \tag{29}$$

$$\text{or} \quad c_0 = \sqrt{\frac{G_{10}}{\rho_0}} \tag{30}$$

Since, the harmonic variation of modulus of rigidity G and density ρ is similar, therefore sound speed is constant i.e. non-homogeneous has no effect on speed and phase of the wave is given $h(x) = \dfrac{x}{c_0}$. So it becomes the case of semi non-homogeneous medium (a medium when characteristics are space dependent while the speed is independent of space variable).

The amplitude function $A_n(x)$ satisfies the equation

$$2h'(x)A'_n(x) + \left\{\rho_0\left(\frac{1}{\eta_{10}} + \frac{1}{\eta_{20}}\right) - 2i\alpha h'(x)\right\} A_n(x) = Q'_n,$$

$$(n = 0,1,2\ldots\ldots\ldots) \tag{31}$$

Where,

$$Q'_n = A''_{n-1} - \left\{\frac{1}{(h')^2} 2i\alpha + 2\frac{G_{20}}{\eta_{20}}\right\} A'_{n-1} + \frac{G_2}{\eta_2}\left\{(h')^2 (2i\alpha) - \frac{G_{10}}{\eta_{10}}(h')^2\right\} A_{n-1} + \frac{G_{20}}{\eta_{20}} A''_{n-2} - \frac{G_{20}}{\eta_{20}}(2i\alpha)A'_{n-2}$$

As the amplitude function is given by Eq.(15), For this case

$$l(x) = \sqrt{\rho_0 G_{10}}\, e^{2i\alpha x}$$

$$m(x) = \frac{\sqrt{\rho_0 G_{10}}}{2}\left(\frac{1}{\eta_{10}} + \frac{1}{\eta_{20}}\right) = m_0$$

$$\int_0^x m(x)dx = m_0 x \tag{32}$$

Hence,

$$A_n(x) = A_n(0)e^{i\alpha x}\exp\{-m_0 x\} \pm \frac{1}{2} c_0\, \mathrm{e}^{i\alpha x} \exp\left\{\pm\int_x^s m_0 dz\right\} Q'^{\pm}_n(s)\, ds \tag{33}$$

For this case the value of first term approximation, the stress function is given by

$$\sigma(x,t)=\sigma_0 e^{i\alpha x} exp\left\{-\int_0^x m_0 ds\right\} H\{t-h(x)\}$$

$$\sigma(x,t)=\sigma^{'}+i\sigma^{''} \tag{34}$$

Where $\sigma^{'}$ and $\sigma^{''}$ represents the real and imaginary parts respectively when $\rho, G, \eta,$ obeys harmonic laws.

The expression for the wave front at $t=h(x)$ is as

$$\sigma(x,t)=\sigma_0(\cos\alpha x) exp\{-m_0 x\}+i\sigma_0(\sin\alpha x) exp\{-m_0 x\} \tag{35}$$

The progressive harmonic wave which starts from the end $x=0$ with amplitude σ_0 and moves with constant velocity $c_0=\sqrt{\frac{G_{10}}{\rho_0}}$ in the positive direction of x is modulated by the factor $\sigma_0(\cos\alpha x) exp\{-m_0 x\}$ and attenuation by the factor $\sigma_0(\sin\alpha x) exp\{-m_0 x\}$.

Case II

$\alpha_1>\alpha_2>\alpha_3$ i.e. density $>$ rigidity $>$ viscosity, then from Eq. (25), we get

$\rho=\rho_0 e^{2i\alpha_1 x}$ $, G=G_0 e^{2i\alpha_2 x}, \eta=\eta_0 e^{2i\alpha_3 x}$

From Eikonal equation of geometric optics

$$\left(\frac{dh(x)}{dx}\right)^2=\frac{\rho}{G_1}=\frac{\rho_0 e^{2i\alpha_1 x}}{G_{10} e^{2i\alpha_2 x}}=\frac{\rho_0}{G_{10}} e^{2i(\alpha_1-\alpha_2)x}=\frac{1}{c^2}\text{ , Here, } c=\sqrt{\frac{G_{10}}{\rho_0}}\, e^{i(\alpha_2-\alpha_1)x} \tag{36}$$

The amplitude function $A_n(x)$ satisfies the equation

$$2h^{'}(x)A^{'}_n(x)+\left\{\rho_0 e^{2i(\alpha_1-\alpha_3)x}\left(\frac{1}{\eta_{10}}+\frac{1}{\eta_{20}}\right)-2i\alpha_1 h^{'}(x)+h''(x)\right\}A_n(x)=Q^{''}_n, (n=0,1,2\ldots\ldots..) \tag{37}$$

Where,

$$Q^{''}_n=A''_{n-1}-\left\{\frac{1}{\left(h^{'}\right)^2}2i\alpha_1+2K\right\}A'_{n-1}-K_1\left\{\left(h^{'}\right)^2(2i\alpha_1)-\frac{G_{10}}{\eta_{10}}e^{2i(\alpha_1-\alpha_3)x}\left(h^{'}\right)^2-2h^{''}\right\}A_{n-1}$$

$$+K_1 A''_{n-2}-2K_1 i\alpha_1 A'_{n-2}(n=0,1,2\ldots\ldots..)$$

and $K_1=e^{2i(\alpha_2-\alpha_3)x}\frac{G_{10}}{G_{20}}$. Amplitude function $A_n(x)$ is given by Eq.(16).

For this case

$$l(x)=\sqrt{G_{10}\rho_0}\, e^{i(\alpha_1+\alpha_2)x}=l_1(x) \text{ and } m(x)=\frac{\sqrt{G_{10}\rho_0}}{2}e^{i(\alpha_1+\alpha_2-2\alpha_3)x}\left\{\frac{1}{\eta_{10}}+\frac{1}{\eta_{20}}\right\}=m_1(x)$$

$$\int_0^x m(x)dx = \frac{\frac{\sqrt{G_{10}\rho_0}}{2}\left\{\frac{1}{\eta_{10}}+\frac{1}{\eta_{20}}\right\}}{i(\alpha_1+\alpha_2-2\alpha_3)}\left\{e^{i(\alpha_1+\alpha_2-2\alpha_3)}-1\right\}, \frac{l(x)}{l(0)} = \left\{e^{i(\alpha_1+\alpha_2)x}\right\} = \frac{l_1(x)}{l_1(0)} \tag{38}$$

Therefore

$$A_n(x) = A_n(0)\left\{e^{i(\alpha_1+\alpha_2)x}\right\}^{\frac{1}{2}} exp\left\{\mp\int_0^x m_1(s)ds\right\} \pm \frac{1}{2}\int_0^x c(s)\left\{\frac{l_1(x)}{l_1(s)}\right\}^{\frac{1}{2}} exp\left\{\pm\int_x^z m_1(z)dz\right\} Q_n^{"\pm}(s)ds \tag{39}$$

For this case the value of first term approximation, the stress function is given by

$$\sigma(x,t) = \sigma_0\left(e^{ir_1x}\right)^{\frac{1}{2}} exp\left\{ip_2\left(e^{ir_2x}-1\right)\right\} H\left\{t-h(x)\right\}$$

$$\sigma(x,t) = \sigma_0\exp(-p_2\sin r_2x)\left(\cos\frac{r_1}{2}x+i\sin\frac{r_1}{2}x\right)\left\{\cos p_2(\cos r_2x-1)+i\sin p_2(\cos r_2x-1)\right\} H\left\{t-h(x)\right\}$$

$$\sigma(x,t) = \sigma' + i\sigma'' \tag{40}$$

The equation of wave front at $t = h(x)$ is given by

$$\begin{aligned}\sigma(x,t) &= \sigma_0\exp(-p_2\sin r_2x)\left[\cos\left\{\frac{r_1}{2}x+p_2(\cos r_2x-1)\right\}\right]\\ &+i\sigma_0\exp(-p_2\sin r_2x)\left[\sin\left\{\frac{r_1}{2}x+p_2(\cos r_2x-1)\right\}\right]\end{aligned} \tag{41}$$

$$\begin{aligned}\sigma(x,t) &= \sigma_0\exp(-p_2\sin r_2x)\left[\cos\left\{\frac{r_1}{2}x+p_2(\cos r_2x-1)\right\}\right]\\ &H\left\{t-h(x)\right\}+i\sigma_0\exp(-p_2\sin r_2x)\left[\sin\left\{\frac{r_1}{2}x+p_2(\cos r_2x-1)\right\}\right]H\left\{t-h(x)\right\}\end{aligned}$$

The progressive harmonic wave which starts from the end $x = 0$ with amplitude σ_0 and moves with constant velocity $c = \sqrt{\frac{G_{10}}{\rho_0}}e^{i(\alpha_2-\alpha_1)x}$ in the positive direction of x is modulated by the factor $\sigma_0\exp(-p_2\sin r_2x)\left[\cos\left\{\frac{r_1}{2}x+p_2(\cos r_2x-1)\right\}\right]$ and attenuation by the factor $\sigma_0\exp(-p_2\sin r_2x)\left[\sin\left\{\frac{r_1}{2}x+p_2(\cos r_2x-1)\right\}\right]$.

Where $r_1 = \alpha_1 + \alpha_2, r_2 = r_1 - 2\alpha_3, \frac{\sqrt{G_{10}\rho_0}}{2}\left\{\frac{1}{\eta_{10}} + \frac{1}{\eta_{20}}\right\} = p_1, \frac{p_1}{r_2} = p_2.$

5. Numerical Analysis

Here, all the mechanical properties obey harmonic laws. As x lies between $0 \le x \le \infty$ and also x depends upon α .Two distinct cases are considered for taking $\alpha < 1$ and $\alpha > 1$.

Let the parameters are as

ρ_0	G_{10}	η_{10}	η_{20}
1.8	1.6	1.2	1.3

Case: 1

For $\alpha < 1$, Let $\alpha = 1/2, m_0 = 8.12$

The equation of wave front at $t = h(x) = 1.06x$ is as

$$\frac{\sigma}{\sigma_0} = (\cos\frac{x}{2}) exp\{-8.12x\} + i(\sin\frac{x}{2}) exp\{-8.12x\} \tag{42}$$

The wave is modulated by the factor

$$(\cos\frac{x}{2}) exp\{-8.12x\} \tag{43}$$

and attenuated by the factor

$$(\sin\frac{x}{2}) exp\{-8.12x\} \tag{44}$$

For $\alpha > 1$, Let $\alpha = 2, m_0 = 8.12$

The equation of wave front at $t = h(x) = 1.06x$ is as

$$\frac{\sigma}{\sigma_0} = (\cos 2x) exp\{-8.12x\} + i\sigma_0(\sin 2x) exp\{-8.12x\} \tag{45}$$

The wave is modulated by the factor

$$(\cos 2x) exp\{-8.12x\} \tag{46}$$

And attenuated by the factor

$$(\sin 2x) exp\{-8.12x\} \tag{47}$$

Case: 2

For non-homogeneous case

	α_1	α_2	α_3	r_1	r_2	p_1	p_2
$\alpha<1$	1/2	1/4	1/6	¾	½	8.12	16.24
$\alpha>1$	8	4	2	12	8	8.12	1.015

For $\alpha<1$

The equation of wave front at $t=h(x)=2.12\sin\dfrac{x}{2}-2.12i\left(\cos\dfrac{x}{2}-1\right)$ is as

$$\frac{\sigma}{\sigma_0}=\exp\left(-16.24\sin\frac{x}{2}\right)\left[\cos\left\{\frac{3}{8}x+16.24\left(\cos\frac{x}{2}-1\right)\right\}\right]+i\exp\left(-16.24\sin\frac{x}{2}\right)\left[\sin\left\{\frac{3}{8}x+16.24\left(\cos\frac{x}{2}-1\right)\right\}\right]$$

The modulated factor is given by

$$\frac{\sigma}{\sigma_0}=\exp\left(-16.24\sin\frac{x}{2}\right)\left[\cos\left\{\frac{3}{8}x+16.24\left(\cos\frac{x}{2}-1\right)\right\}\right] \tag{48}$$

The attenuation is given by the factor

$$\exp\left(-16.24\sin\frac{x}{2}\right)\left[\sin\left\{\frac{3}{8}x+16.24\left(\cos\frac{x}{2}-1\right)\right\}\right] \tag{49}$$

For $\alpha>1$

The equation of wave front at $t=h(x)=0.26\sin\dfrac{x}{2}-i(0.26)\left(\cos 4x-1\right)$ is as

$$\begin{aligned}\frac{\sigma}{\sigma_0}&=\exp\left(-1.015\sin 8x\right)\left[\cos\left\{6x+(1.015)\left(\cos 8x-1\right)\right\}\right]\\&+i\exp\left(-1.015\sin 8x\right)\left[\sin\left\{6x+1.015\left(\cos 8x-1\right)\right\}\right]\end{aligned} \tag{50}$$

The modulated factor is given by

$$\exp\left(-1.015\sin 8x\right)\left[\cos\left\{6x+(1.015)\left(\cos 8x-1\right)\right\}\right] \tag{51}$$

The attenuation is given by the factor

$$\exp\left(-1.015\sin 8x\right)\left[\sin\left\{6x+1.015\left(\cos 8x-1\right)\right\}\right] \tag{52}$$

To see qualitative effect of non-homogeneity on the harmonic wave propagation in non-homogeneous four parameter viscoelastic rods, the various graphs are plotted between $\frac{\sigma}{\sigma_0}$ and x. For the semi homogeneous cases Fig.2 represents the plot for Eq.(43) and Fig.(3) represents the plot for Eq.(44).It shows that there is slight variation in the wave in the neighborhood of $x = 0$ As x increases the wave becomes constant. Fig.(2) represents the wave in progress and fig.(3) represents its attenuation. It is also observed that for $\alpha < 1$,the wave progression is near the origin and not along the x apart. The Fig.(4) and (5) also represents the similar result for $\alpha > 1$. So it can be concluded that the value of α does not impact more in the semi homogeneous case and also that the wave progression is at near the starting point only. For the non-homogeneous cases, Fig.(6) and Fig.(7) represents the plot for Eq.(48) and Eq.(49) respectively. Fig.(8) and Fig.(9) represents the plot for Eq.(51) and Eq.(52) respectively For $\alpha < 1$ and for $\alpha > 1$ harmonic wave is in progress but with unequal interval of times. Thus the effect of non-homogeneity is clearly observed for the harmonic waves in non-homogeneous four parameter viscoelastic model.

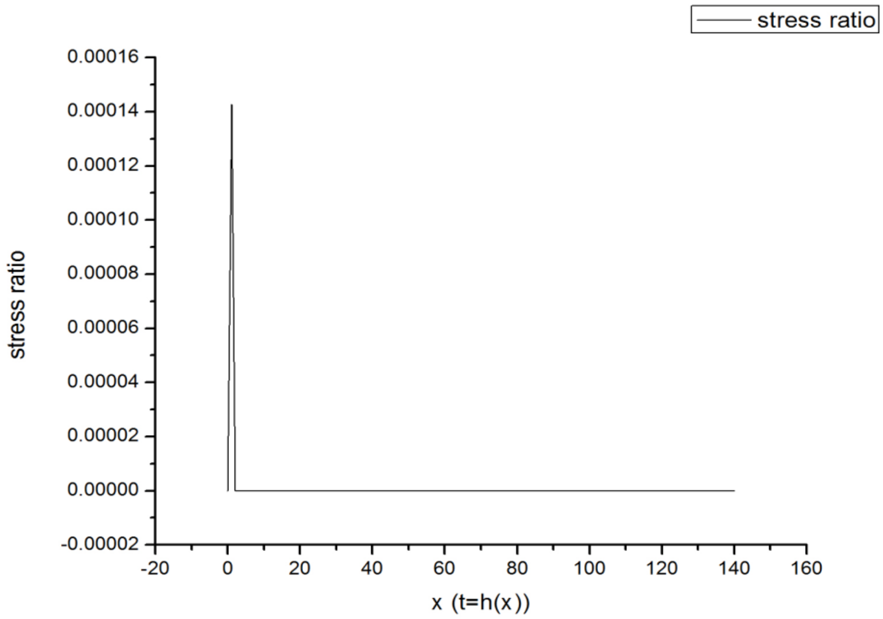


Fig. 2

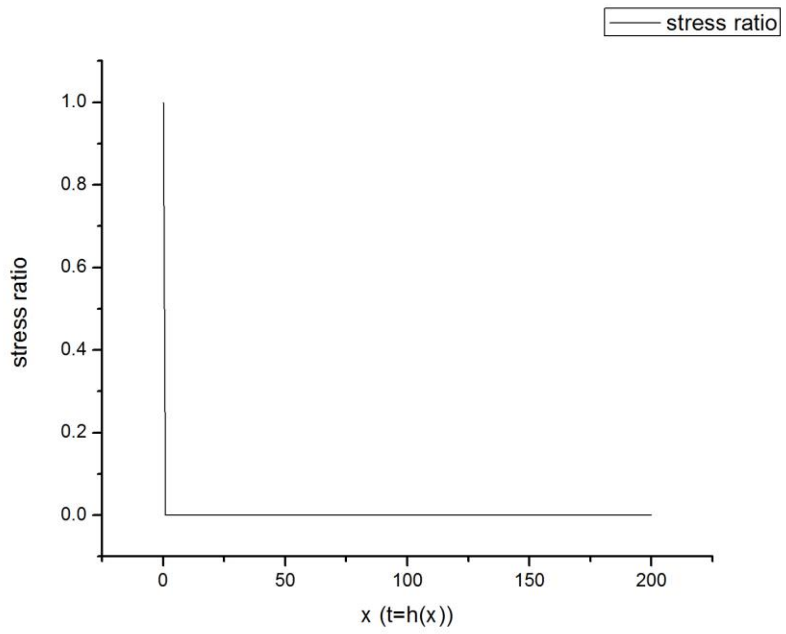

stress ratio
1.0
0.8
0.6
0.4
0.2
0.0
stress ratio
0
50
100
150
200
x (t=h(x))

Fig. 3

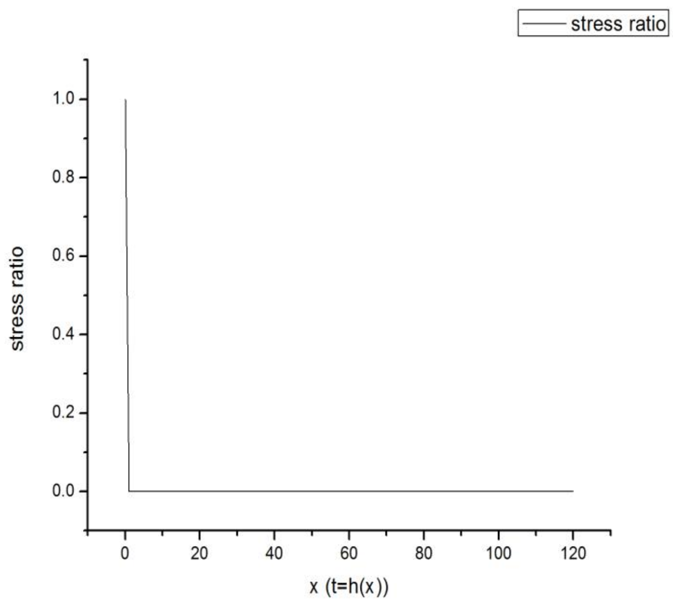
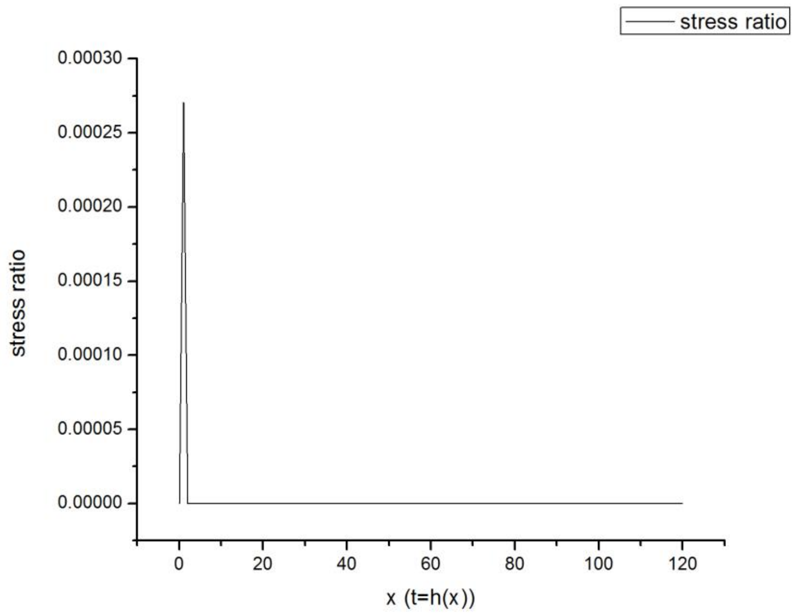
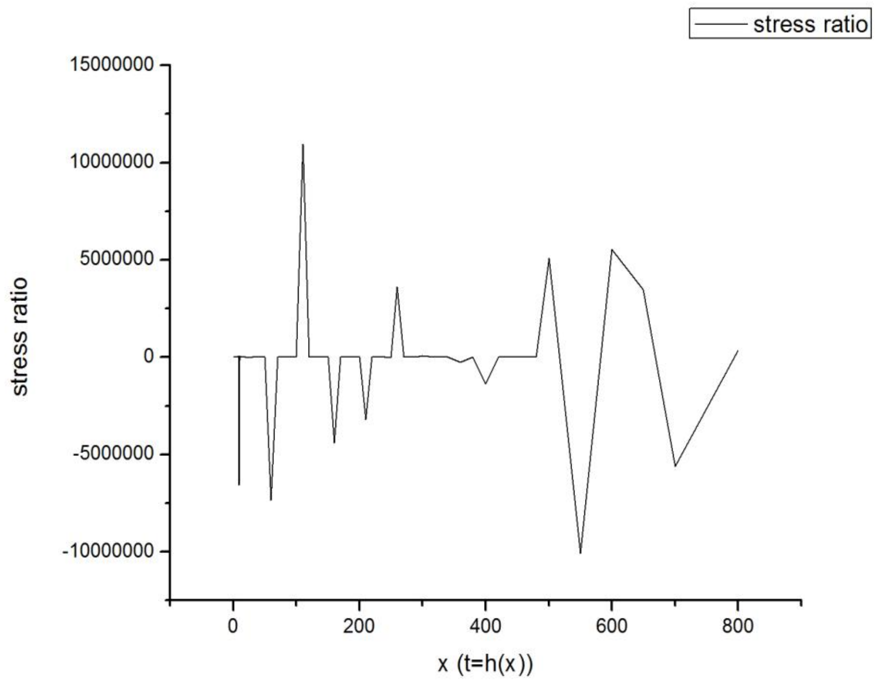

stress ratio
1.0
0.8
0.6
0.4
0.2
0.0
stress ratio
0
20
40
60
80
100
120
x (t=h(x))

Fig. 4



stress ratio
0.00030
0.00025
0.00020
0.00015
0.00010
0.00005
0.00000
stress ratio
0
20
40
60
80
100
120
x (t=h(x))

Fig. 5



stress ratio
15000000
10000000
5000000
0
-5000000
-10000000
stress ratio
0
200
400
600
800
x (t=h(x))

Fig. 6

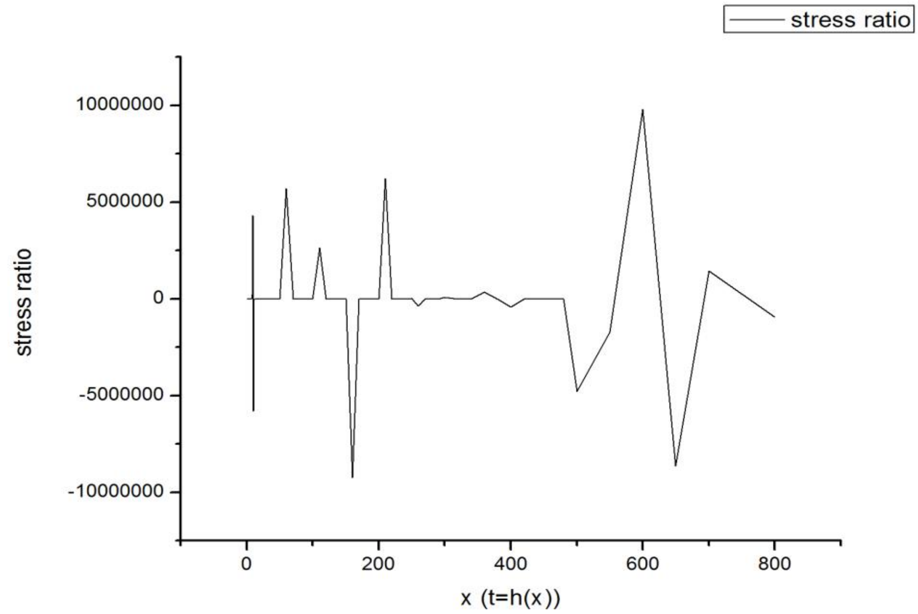

stress ratio
10000000
5000000
0
-5000000
-10000000
stress ratio
0
200
400
600
800
x (t=h(x))

Fig. 7

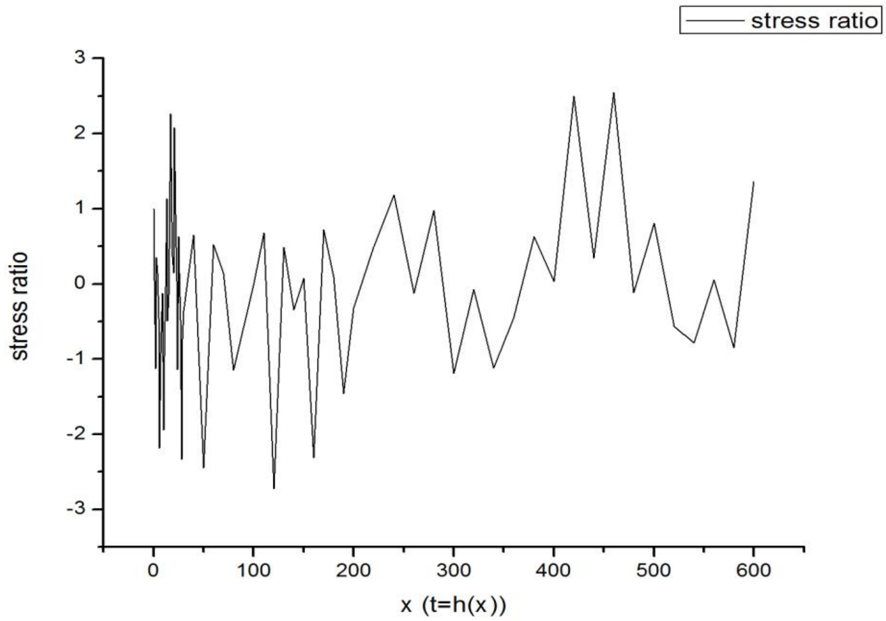

stress ratio
3
2
1
0
-1
-2
-3
stress ratio
0
100
200
300
400
500
600
x (t=h(x))

Fig. 8

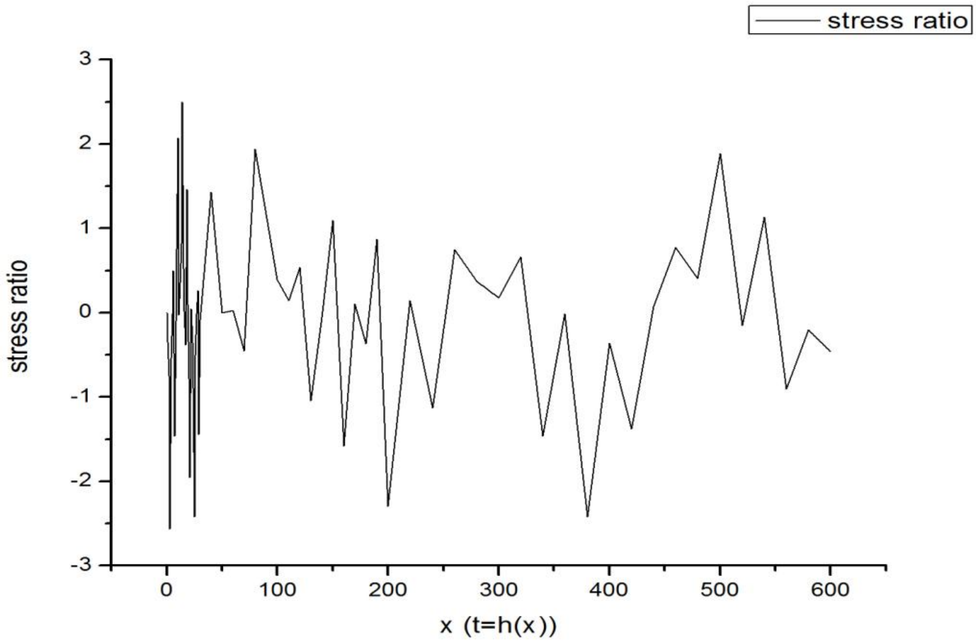


Fig. 9

6. Conclusions

1. When the density, rigidity and viscosity all are equal for the first material specimen, the sound speed is constant i.e. non-homogeneous has no effect on speed and phase of the wave is given $h(x) = \dfrac{x}{c_0}$. So it becomes the case of semi non-homogeneous medium (a medium when characteristics are space dependent while the speed is independent of space variable). The longitudinal speed will be equal to $c = \sqrt{\dfrac{G_{10}}{\rho_0}}$
2. When the density, rigidity and viscosity are not equal for the second material specimen, the speed of sound varies exponential as $c = \sqrt{\dfrac{G_{10}}{\rho_0}} e^{i(\alpha_2 - \alpha_1)x}$

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