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The embedding theorems of space $W_{p,\varphi,\beta}^{l}(G)$

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Abstract

In this paper a generalized Sobolev-Morrey spaces $W_{p,\varphi,\beta}^{l}(G)$ is introduced. With the help of Sobolev integral representation is obtained embedding theorems.

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1 Introduction and Preliminary Notes

In the paper, we introduce a space with Sobolev-Morrey type parameters denoted by $W_{p,\varphi,\beta}^{l}(G)$. The space $W_{p,\varphi,\beta}^{l}(G)$ under consideration consists of the set of locally summable on G functions f having on G the generalized derivatives $D_i^{l_i} f$ ($l_i > 0$ are entire, $i = 1, 2, ..., n$) with the finite norm

$$\|f\|_{W_{p,\varphi,\beta}^{l}(G)} = \|f\|_{p,\varphi,\beta;G} + \sum_{i=1}^{n} \left\|D_i^{l_i} f\right\|_{p,\varphi,\beta;G}, \tag{1}$$

where

$$\|f\|_{p,\varphi,\beta;G} = \|f\|_{L_{p,\varphi,\beta}(G)} = \sup_{\substack{x\in G,\\ t>0}} \left(|\varphi([t]_1)|^{-\beta} \|f\|_{p,G_{\varphi(t)}(x)} \right), \tag{2}$$

for any $x \in R^n$,

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t), \quad j = 1, 2, ..., n \right\},$$

$p\in[1,\infty)$, $\varphi(t)=(\varphi_1(t),...,\varphi_n(t))$, $\varphi_j(t)>0\quad(t>0)$ by Lebesgue measurable $\lim\limits_{t\to+0}\varphi_j(t)=0$, $\lim\limits_{t\to+\infty}\varphi_j(t)=\infty, |\varphi([t]_1)|^{-\beta}=\prod\limits_{j=1}^{n}(\varphi_j([t]_1))^{-\beta_j}$, $[t]_1=\min\{1,t\}$, $\beta_j\in[0,1]$, $j=1,2,...,n$. Denote the set of such vectors by N.

Let for any $t>0$ $|\varphi([t]_1)|\le C$, where C is some positive constant. Then the embeddings $L_{p,\varphi,\beta}(G)\to L_p(G)$ and $W^l_{p,\varphi,\beta}(G)\to W^l_p(G)$ hold, i.e.

$$\|f\|_{p,G}\le C\|f\|_{p,\varphi,\beta;G},\quad \|f\|_{W^l_p(G)}\le C\|f\|_{W^l_{p,\varphi,\beta}(G)}.$$

Note that the spaces $L_{p,\varphi,\beta}(G)$ and $W^l_{p,\varphi,\beta}(G)$ are Banach spaces. The completeness of these spaces automatically implies from completeness of L_p and W^l_p. The space $W^l_{p,\varphi,\beta}(G)$, when $\varphi_j(t)=t^{\chi_j}$, $\beta_j=\frac{a_j}{p}$ $(j=1,...,n)$ coincides with the space $W^l_{p,a,\chi}(G)$ introduced by V.P. Il'yin [9], in the case $\beta_j=0\,(j=1,...,n)$ it coincides with the Sobolev space $W^l_p(G)$. The spaces of such type with different norms were introduced and studied in [2]-[8] and [10], [11].

Definition 1.1 *The open set $G\subset R^n$ is said to be an open set with condition of flexible φ-horn if for some $\theta\in(0,1]^n$, $T\in(0,\infty)$ for any $x\in G$ there exists the vector-function*

$$\rho(\varphi(t),x)=(\rho_1(\varphi_1(t),x),...,\rho_n(\varphi_n(t),x)),\quad 0\le t\le T$$

with the following properties:

1) for all $j=1,2,...,n$, $\rho(\varphi_j(t),x)$ is absolutely continuous on $[0,T]$, $\left|\rho'_j(\varphi_j(t),x)\right|\le 1$ for almost all $t\in[0,T]$,

2) $\rho_j(0,x)=0$, $x+\bigcup\limits_{0\le t\le T}[\rho(\varphi(t),x)+\varphi(t)\theta I]\subset G$.

In particular, $\varphi(t)=t^\lambda$, $\left(t^\lambda=\left(t^{\lambda_1},t^{\lambda_2},...,t^{\lambda_n}\right)\right)$ is the set V and $x+V$ will be said to be a set of flexible λ-horn introduced in [1].

Assuming that $\varphi_j(t)\ (j=1,2,...,n)$ are also differentiable on $[0,T]$, we can show that for $f\in W^l_p(G)$ determined in n- dimensional domains, satisfying the condition of flexible φ-horn, it holds the following integral representation $(\forall x\in U\subset G)$

$$D^\nu f(x)=f^{(\nu)}_{\varphi(T)}(x)+\sum_{i=1}^{n}\int_0^T\int_{R^n}L_i^{(\nu)}\left(\frac{y}{\varphi(t)},\frac{\rho(\varphi(t,x))}{\varphi(t)},\rho'(\varphi(t),x)\right)\times$$

$$\times D_i^{l_i}f(x+y)\prod_{j=1}^{n}(\varphi_j(t))^{l_j-\nu_j-1}\frac{\varphi'_i(t)}{\varphi_i(t)}dtdy, \tag{3}$$

$$f^{(\nu)}_{\varphi(T)}(x)=(-1)^{|\nu|}\prod_{j=1}^{n}(\varphi_j(T))^{-1-\nu_j}\int_{R^n} f(x+y)\,\Omega^{(\nu)}\left(\frac{y}{\varphi(T)},\frac{\rho(\varphi(T),x)}{\varphi(T)}\right)dy. \quad (4)$$

Let $M_0(\cdot,y,z)\in C_0^\infty(R^n)$ be such that

$$S(M)\subset I_{\varphi(T)}=\left\{y:|y_j|<\frac{1}{2}\varphi_j(T),\quad j=1,2,...,n\right\}.$$

Assume that for any $0<T\le 1$ (T_0 is a fixed number)

$$V=\bigcup_{0<t\le T}\left\{y:\frac{y}{\varphi(t)}\in S(M)\right\}.$$

It is clear that $V\subset I_{\varphi(T)}$ and suppose that $U+V\subset G$.

Lemma 1.2 *Let* $1\le p\le q\le r\le\infty$; $0<\eta,\ t<T\le 1$, $\nu=(\nu_1,\nu_2,...,\nu_n)$, $\nu_j\ge 0$ *be entire,* $j=1,2,...,n$; $\Phi\in L_{p,\varphi,\beta}(G)$

$$A^i_\eta(x)=\int_0^T\int_{R^n} M\left(\frac{y}{\varphi(t)},\frac{\rho(\varphi(t),x)}{\varphi(t)},\rho'(\varphi(t),x)\right)\times$$

$$\times\Phi(x+y)\prod_{j=1}^{n}(\varphi_j(t))^{l_j-\nu_j-1}\frac{\varphi_i'(t)}{\varphi_i(t)}dtdy \quad (5)$$

$$A^i_{\eta,T}(x)=\int_0^T\int_{R^n} M\left(\frac{y}{\varphi(t)},\frac{\rho(\varphi(t),x)}{\varphi(t)},\rho'(\varphi(t),x)\right)\times$$

$$\times\Phi(x+y)\prod_{j=1}^{n}(\varphi_j(t))^{l_j-\nu_j-1}\frac{\varphi_i'(t)}{\varphi_i(t)}dtdy \quad (6)$$

and let

$$Q^i_T=\int_0^T\prod_{j=1}^{n}(\varphi_j(t))^{l_j-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)}\frac{\varphi_i'(t)}{\varphi_i(t)}dt<\infty. \quad (7)$$

Then for any $\overline{x}\in U$ *the following inequalities are true*

$$\sup_{\overline{x}\in U}\left\|A^i_\eta\right\|_{q,U_{\psi(\xi)}(\overline{x})}\le C_1\|\Phi\|_{p,\varphi,\beta;G}\left|Q^i_\eta\right|\prod_{j=1}^{n}(\psi_j([\xi]_1))^{\beta_j\frac{p}{q}}, \quad (8)$$

$$\sup_{\overline{x}\in U}\left\|A^i_{\eta,T}\right\|_{q,U_{\psi(\xi)}(\overline{x})}\le C_2\|\Phi\|_{p,\varphi,\beta;G}\left|Q^i_{\eta,T}\right|\prod_{j=1}^{n}(\psi_j([\xi]_1))^{\beta_j\frac{p}{q}}, \quad (9)$$

where $U_{\psi(\xi)}(\overline{x})=\left\{x:|x_j-\overline{x}_j|<\frac{1}{2}\psi_j(\xi),j=1,2,...,n\right\}$ *and* $\psi\in N$, C_1, C_2 *are the constants independent of* φ, ξ, η *and* T.

Proof. Applying sequentially the Minkowsky generalized inequality for any $\overline{x} \in U$

$$\left\|A_\eta^i\right\|_{q,U_{\psi(\xi)}(\overline{x})} \leq \int_0^\eta \|F(\cdot,t)\|_{q,U_{\psi(\xi)}(\overline{x})} \prod_{j=1}^n (\varphi_j(t))^{l_j-\nu_j-1} \frac{\varphi_i'(t)}{\varphi_i(t)} dt, \tag{10}$$

where

$$F(x,t) = \int_{R^n} M\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}, \rho'(\varphi(t),x)\right)\Phi(x+y)\,dy \qquad (11).$$

From the Holder inequality ($q \leq r$) we have

$$\|F(\cdot,t)\|_{q,U_{\psi(\xi)}(\overline{x})} \leq \|F(\cdot,t)\|_{r,U_{\psi(\xi)}(\overline{x})} \prod_{j=1}^n (\psi_j(\xi))^{\frac{1}{q}-\frac{1}{r}}. \tag{12}$$

Now estimate the norm $\|F(\cdot,t)\|_{q,U_{\psi(\xi)}(\overline{x})}$. Let X be a characteristic function of the set $S(M) = supp\, M$. Noting that $1 \leq p \leq r \leq \infty$, $s \leq r$, represent the integrand function (11) in the form

$$|M\Phi| = (|\Phi|^p |M|^s)^{\frac{1}{r}} (|\Phi|^p X)^{\frac{1}{q}-\frac{1}{r}} (|M|^s)^{\frac{1}{s}-\frac{1}{r}}$$

and apply for $|F|$ the Holder inequality $\left(\frac{1}{r} + \left(\frac{1}{p} - \frac{1}{r}\right) + \left(\frac{1}{s} - \frac{1}{r}\right) = 1\right)$, we obtain

$$\|F(\cdot,t)\|_{r,U_{\psi(\xi)}(\overline{x})} \leq \sup_{x \in U_{\psi(\xi)}(\overline{x})} \times$$

$$\times \left(\int_{R^n} |\Phi(x-y)|^p \chi\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}\right) dy\right)^{\frac{1}{p}-\frac{1}{r}} \times$$

$$\times \sup_{y \in V} \left(\int_{U_{\psi(\xi)}(\overline{x})} |\Phi(x+y)|^p\,dx\right)^{\frac{1}{r}} \times$$

$$\times \left(\int_{R^n} \left|M\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}, \rho'(\varphi(t),x)\right)\right|^s dy\right)^{\frac{1}{s}}. \tag{13}$$

For any $x \in U$ we have

$$\int_{R^n} |\Phi(x+y)|^p \chi\left(\frac{y}{\varphi(t)}\right) dy \leq \int_{(U+V)_{\varphi(t)}(\overline{x})} |\Phi(y)|^p\,dy \leq$$

$$\leq \int\limits_{G_{\varphi(t)}(\overline{x})} |\Phi(y)|^p \, dy \leq \|\Phi\|^p_{p,\varphi,\beta;G} \cdot \prod_{j=1}^{n} (\varphi_j(t))^{\beta_j p} \qquad (14).$$

For $y \in V$

$$\int\limits_{U_{\psi(\xi)}(\overline{x})} |\Phi(x+y)|^p \, dx \leq \int\limits_{G_{\psi(\xi)}(\overline{x}+y)} |\Phi(x)|^p \, dx \leq \|\Phi\|^p_{p,\psi,\beta;G} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\beta_j p} \quad (15)$$

$$\int\limits_{R^n} \left| M\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \rho'(\varphi(t), x)\right)\right|^s dy = \|M\|_s \cdot \prod_{j=1} \varphi_j(t). \qquad (16)$$

From inequalities (13)-(16) it follows that

$$\|F(\cdot, t)\|_{r, U_{\psi(\xi)}(\overline{x})} \leq \|M\|_s \cdot \|\Phi\|_{p,\varphi,\beta;G} \times$$

$$\times \prod_{j=1}^{n} (\varphi_j(t))^{\frac{1}{s} + \beta_j p \left(\frac{1}{p} - \frac{1}{r}\right)} \cdot \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\frac{\beta_j p}{r}}. \qquad (17)$$

Inequalities (10), (12) and (17) for $(r = q)$ and for any $\overline{x} \in U$ reduce to the estimation

$$\left\|A^i_\eta\right\|_{r, U_{\psi(\xi)}(\overline{x})} \leq C_1 \|\Phi\|_{p,\varphi,\beta;G} \left|Q^i_\eta\right| \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\beta_j \frac{p}{q}} \quad \left(Q^i_\eta < \infty\right). \qquad (18)$$

In the case $Q^i_{\eta,T} < \infty$ inequality (9) is proved in the same way.

From inequalities (17) and (18) for $(r = q)$ we get the inequality $(\forall \overline{x} \in U)$

$$\sup_{\overline{x} \in U} \|F\|_{q, U_{\psi(\xi)}(\overline{x})} \leq C_3 \|\Phi\|_{p,\varphi,\beta;G} \cdot \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\beta_j \frac{p}{q}},$$

$$\sup_{\overline{x} \in U} \left\|A^i_\eta\right\|_{q, U_{\psi(\xi)}(\overline{x})} \leq C_3 \|\Phi\|_{p,\varphi,\beta;G} \cdot \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\beta_j \frac{p}{q}}.$$

From last inequalities it follows that

$$\|F\|_{q,\psi,\beta^1;U} \leq C'_1 \|\Phi\|_{p,\varphi,\beta;G}, \qquad (19)$$

$$\left\|A^i_\eta\right\|_{q,\psi,\beta^1;U} \leq C'_2 \|\Phi\|_{p,\varphi,\beta;G}. \qquad (20)$$

C'_1 and C'_2 are the constants independent of Φ.

2 Main Results

Prove two theorems on the properties of the functions from the space $W^l_{p,\varphi,\beta}(G)$.

Theorem 2.1 *Let $G \subset R^n$ satisfy the condition of flexible φ-horn, $1 \leq p \leq q \leq \infty$, $\nu = (\nu_1, \nu_2, .., \nu_n)$, $\nu_j \geq 0$ be entire $j = 1, 2, ..., n$, $Q^i_T < \infty$ $(i = 1, 2, ..., n)$ and let $f \in W^l_{p,\varphi,\beta}(G)$. Then the following embeddings hold*

$$D^\nu : W^l_{p,\varphi,\beta}(G) \to L_{q,\psi,\beta^1}(G) \ \text{and} \ D^\nu : W^l_{p,\varphi,\beta}(G) \to W^{l^1}_{q,\psi,\beta^1}(G),$$

i.e. for $f \in W^l_{p,\varphi,\beta}(G)$ there exists a generalized derivative $D^\nu f$ and the following inequalities are true

$$\|D^\nu f\|_{q,G} \leq C_1 \left(B_1(T) \|F\|_{q,\psi,\beta;G} + \sum_{i=1}^n \left|Q^i_T\right| \left\|D^{l_i}_i f\right\|_{p,\varphi,\beta;G} \right), \tag{21}$$

$$\|D^\nu f\|_{q,\psi,\beta^1;G} \leq C_2 \|f\|_{W^l_{p,\varphi,\beta}(G)}, \ p \leq q < \infty, \tag{22}$$

and if

$$Q^{i,j}_T = \int_0^T \prod_{j=1}^n (\varphi_j(t))^{l_j - \nu_j - (1-\beta_j p)\left(\frac{1}{p} - \frac{1}{q}\right) - l^1_j} \frac{\varphi'_i(t)}{\varphi_i(t)} dt < \infty,$$

then

$$\|D^\nu f\|_{W^l_q(G)} \leq C_1 \left(B_2(T) \|f\|_{p,\varphi,\beta;G} + \sum_{i=1}^n \left|Q^i_T\right| \left\|D^{l_i}_i f\right\|_{p,\varphi,\beta;G} \right) \tag{23}$$

$$\|D^\nu f\|_{W^{l^1}_{q,\psi,\beta}(G)} \leq \|f\|_{W^l_{p,\varphi,\beta}(G)}, \quad p \leq q < \infty. \tag{24}$$

In particular, if

$$\int_0^T \prod_{j=1}^n (\varphi_j(t))^{l_j - \nu_j - (1-\beta_j p)\frac{1}{p}} \frac{\varphi'_i(t)}{\varphi_i(t)} dt < \infty, \tag{25}$$

then $D^\nu f(x)$ is continuous on G, i.e.

$$\sup_{x \in G} |D^\nu f(x)| \leq C_1 \left(B_1(T) \|f\|_{p,\varphi,\beta;G} + \sum_{i=1}^n \left|Q^i_T\right| \left\|D^{l_i}_i f\right\|_{p,\varphi,\beta;G} \right) \tag{26}$$

$0 < T \leq \min\{1, T_0\}$, *$T_0$ is a fixed number; C_1, C_2, C_3, C_4 are the constants independent of f, C_1 and C_3 are independent also on T.*

Proof. At first note that in the conditions of our theorem there exists a generalized derivative $D^\nu f$ on G. Indeed, from the condition $Q^i_T < \infty$ for all $(i = 1, 2, ..., n)$ it follows that for $f \in W^l_{p,\varphi,\beta}(G) \to W^l_p(G)$, there exists $D^\nu f \in L_p(G)$ and for it integral representation (3) and (4) with the same kernels is valid.

Based around the Minkowsky inequality, from identities (3) and (4) we get

$$\|D^\nu f\|_{q,G} \le \left\| f^{(\nu)}_{\varphi(T)} \right\|_{q,G} + \sum_{i=1}^{n} \left\| A^i_T \right\|_{q,G}. \tag{27}$$

By means of inequality (17) for $U = G$, $\Phi = f$ we get

$$\left\| f^{(\nu)}_{\varphi(T)} \right\|_{q,G} \le \|f\|_{p,\varphi,\beta;G} \prod_{j=1}^{n} (\varphi_j(T))^{-\nu_j - (1-\beta_j p)\left(\frac{1}{p} - \frac{1}{q}\right)} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\beta_j \frac{p}{q}} \le$$

$$\le C_1 B(T) \|f\|_{p,\varphi,\beta;G}, \tag{28}$$

and from inequality (8) for $\eta = T$, $U = G$, $\Phi = D^{l_i}_i f$ we get

$$\left\| A^i_T \right\|_{q,G} \le C_2 |Q^i_T| \left\| D^{l_i}_i f \right\|_{p,\varphi,\beta;G}. \tag{29}$$

Substituting (28) and (29) in (27), we get inequality (21). By means of inequalities (19) and (20) for $\eta = T$ we get inequality (22).

In identities (3) and (4) instead of ν_j we take $\nu_j + l^1_j$ ($l^1_j > 0$ are entire, $j = 1, 2, ..., n$) for $U = G$ and inequalities (23) and (24) are proved in the same way.

Now let conditions (25) be satisfied, then based around identities (3), (4), from inequality (27) we get

$$\left\| D^\nu f - f^{(\nu)}_{\varphi(T)} \right\|_{\infty,G} \le C \sum_{i=1}^{n} |Q^i_T| \left\| D^{l_i}_i f \right\|_{p,\varphi,\beta;G}.$$

As $T \to 0$, the left side of this inequality tends to zero, since $f^{(\nu)}_{\varphi(T)}(x)$ is continuous on G and the convergence on $L_\infty(G)$ coincides with the uniform convergence. Then the limit function $D^\nu f$ is continuous on G.

Theorem 1 is proved.

Let γ be an n-dimensional vector.

Theorem 2.2 *Let all the conditions of theorem 1 be fulfilled. Then for* $Q^i_T < \infty$ $(i = 1, 2, ..., n)$ *the derivative* $D^\nu f$ *satisfies on* G *the Holder generalized condition, i.e. the following inequality is valid:*

$$\|\Delta(\gamma, G) D^\nu f\|_{q,G} \le C \|f\|_{W^l_{p,\varphi,\beta}(G)} \cdot |h(|\gamma|, \varphi; T)|, \tag{30}$$

where C *is a constant independent of* f, $|\gamma|$ *and* T.

Proof. According to lemma 8.6 from [1] there exists a domain

$$G_\omega \subset G\left(\omega = \zeta r\left(x\right),\ \zeta > 0\ r\left(x\right) = \rho\left(x, \partial G\right),\ x \in G\right)$$

and assume that $|\gamma| < \omega$, then for any $x \in G_\omega$ the segment connecting the points $x, x+\gamma$ is contained in G. Consequently, for all the points of this segment, identies (3), (4) with the same kernels are valid. After same transformations, from (3) and (4) we get

$$\left|\Delta\left(\gamma, G\right) D^\nu f\left(x\right)\right| \leq \prod_{j=1}^{n}\left(\varphi_j\left(T\right)\right)^{-1-\nu_j} \times$$

$$\times \int\limits_{R^n}\left|f\left(x+y\right)\right|\left|\Omega^{(\nu)}\left(\frac{y-\gamma}{\varphi\left(T\right)}, \frac{\rho\left(\varphi\left(T\right), x\right)}{\varphi\left(T\right)}\right)-\right.$$

$$\left.-\ \Omega^{(\nu)}\left(\frac{y}{\varphi\left(T\right)}, \frac{\rho\left(\varphi\left(T\right), x\right)}{\varphi\left(T\right)}\right)\right| dy +$$

$$+\sum_{i=1}^{n}\left\{\int\limits_{0}^{|\gamma|}\int\limits_{R^n}\left(\left|D_i^{l_i} f\left(x+y+\gamma\right)\right| + \left|D_i^{l_i} f\left(x+y\right)\right|\right) \times\right.$$

$$\times \left|L_i^{(\nu)}\left(\frac{y}{\varphi\left(T\right)}, \frac{\rho\left(\varphi\left(T\right), x\right)}{\varphi\left(T\right)}, \rho'\left(\varphi\left(t\right), x\right)\right)\right| \prod_{j=1}^{n}\left(\varphi_j\left(t\right)\right)^{l_j-\nu_j-1}\frac{\varphi_i'\left(t\right)}{\varphi_i\left(t\right)} dy dt +$$

$$+\int\limits_{|\gamma|}^{T}\int\limits_{R^n}\left|D_i^{l_i} f\left(x+y\right)\right| \times \left|L_i^{(\nu)}\left(\frac{y-\gamma}{\varphi\left(t\right)}, \frac{\rho\left(\varphi\left(t\right), x\right)}{\varphi\left(t\right)}, \rho'\left(\varphi\left(t\right), x\right)\right)-\right.$$

$$\left.\left.-L_i^{(\nu)}\left(\frac{y}{\varphi\left(t\right)}, \frac{\rho\left(\varphi\left(t\right), x\right)}{\varphi\left(t\right)}, \rho'\left(\varphi\left(t\right), x\right)\right)\right| \prod_{j=1}^{n}\left(\varphi_j\left(t\right)\right)^{l_j-\nu_j-1}\frac{\varphi_i'\left(t\right)}{\varphi_i\left(t\right)} dy dt\right\} =$$

$$= E\left(x, \gamma\right) + \sum_{i=1}^{n}\left(E_{|\gamma|}^{i}\left(x, \gamma\right) + E_{|\gamma|, T}^{i, T}\left(x, \gamma\right)\right), \tag{31}$$

where $0 < T \leq \{1, T_0\}$ we also assume that $|\gamma| < T$. Consequently, $|\gamma| < \min\left(\omega, T\right)$. If $x \in G \setminus G_\omega$ then by definition

$$\Delta\left(\gamma, G\right) D^\nu f\left(x\right) = 0.$$

Based around (31) we have

$$\left\|\Delta\left(\gamma, G\right) D^\nu f\right\|_{q,G} \leq \left\|E\left(\cdot, \gamma\right)\right\|_{q,G_\omega} +$$

$$+\sum_{i=1}^{n}\left(\left\|E^i_{|\gamma|}(\cdot,\gamma)\right\|_{q,G_\omega}+\left\|E^i_{|\gamma|,T}(\cdot,\gamma)\right\|_{q,G_\omega}\right), \tag{32}$$

$$E(x,\gamma)\le\prod_{j=1}^{n}(\varphi_j(T))^{-\nu_j-2}\int_0^{|\gamma|}d\zeta\times$$

$$\times\int_{R^n}|f(x+\zeta e_\gamma+y)|\,|D_j\Omega^{(\nu)}\left(\frac{y}{\varphi(T)},\frac{\rho(\varphi(T),x)}{\varphi(T)}\right)dy \tag{33}$$

where $e_\gamma=\dfrac{\gamma}{|\gamma|}$.

Similarly we get

$$E^i_{|\gamma|}\le c_2\int_0^{|\gamma|}\prod_{j=1}^{n}(\varphi_j(t))^{l_j-\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dt\int_{R^n}\left|D_i^{l_i}f(x+\zeta e_i+y)\right|\times$$

$$\times\left|D_jL_i^{(\nu)}\left(\frac{y}{\varphi(t)},\frac{\rho(\varphi(t),x)}{\varphi(t)},\rho'(\varphi(t),x)\right)\right|dy. \tag{34}$$

Taking into account $\xi e_\gamma+G_\omega\subset G$, based around the generalized Minkowsky inequality, from inequality (31) and from inequality (17) for $U=G$, $f=\Phi$, $M=\Omega^{(\nu)}$ we have

$$\|E(\cdot,\gamma)\|_{q,G_\omega}\le C_1|\gamma|\,\|f\|_{p,\varphi,\beta;G} \tag{35}$$

By means of inequality (8), for $U=G$, $D_i^{l_i}f=\Phi$, $M=L_i^{(\nu)}$, $\eta=|\gamma|$ we get

$$\left\|E^i_{|\gamma|}(\cdot,\gamma)\right\|_{q,G_\omega}\le C_2\left|Q^i_{|\gamma|}\right|\left\|D_i^{l_i}f\right\|_{p,\varphi,\beta;G} \tag{36}$$

ad by means of inequality (9) for $U=G$, $D_i^{l_i}f=\Phi$, $M=L_i^{(\nu)}$, $\eta=|\gamma|$ we get

$$\left\|E^i_{|\gamma|,T}(\cdot,\gamma)\right\|_{q,G_\omega}\le C_3\left|Q^i_{|\gamma|,T}\right|\left\|D_i^{l_i}f\right\|_{p,\varphi,\beta;G} \quad (37).$$

From inequalities (32) and (35)-(37) we get the required inequality.

Now suppose that $|\gamma|\ge\min(\omega,T)$. Then

$$\|\Delta(\gamma,G)D^\nu f\|_{q,G}\le2\|D^\nu f\|_{q,G}\le C(\omega T)\|D^\nu f\|_{q,G}|h(|\gamma|,\varphi;T)|.$$

Estimating for $\|D^\nu f\|_{q,G}$ by means of inequality (21), in this case we get estimation (30).

Theorem 2 is proved.

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