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Volterra composition operators between weighted Bergman and Bloch-type spaces

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Abstract

In this paper, we characterize boundedness and compactness of the volterra composition operators between weighted Bergman and Bloch spaces.

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1 Introduction

Let $\mathbf{D}$ be the open unit disk in the complex plane $\mathbf{C}$, $H(\mathbf{D})$ be the space of holomorphic functions on $\mathbf{D}$. Let $dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta$ be the normalized area measure on $\mathbf{D}$. Recall that (see, for example [4]) positive continuous function ω on $\mathbf{D}$ is a normal weight if

(i) ω is a radial weight, that is, $\omega(z) = \omega(|z|)$ for every $z \in \mathbf{D}$.

(ii) there exist positive numbers s and t, $0 < s < t$ such that

$$\frac{\omega(r)}{(1-r)^s} \to 0, \quad \frac{\omega(r)}{(1-r)^t} \to \infty, \text{ as } r \to 1^-.$$

For $0 < p < \infty$, and a normal weight function ω, let $\mathcal{A}(p,\omega)$ denote the space of all holomorphic functions f on $\mathbf{D}$ such that

$$||f||_{\mathcal{A}(p,\omega)} = \int_{\mathbf{D}} |f(z)|^p \frac{\omega^p(|z|)}{1-|z|} dA(z) < \infty.$$

For $1 \le p < \infty$, $\mathcal{A}(p,\omega)$ is a Banach space equipped with the norm $||\cdot||_{\mathcal{A}(p,\omega)}$. When $0 < p < 1$, $||\cdot||_{\mathcal{A}(p,\omega)}$ is a quasinorm on $\mathcal{A}(p,\omega)$ and $\mathcal{A}(p,\omega)$ is a Frechet space, but not a Banach space. Note that if $\omega(r) = (1-r)^{1/p}$, then $\mathcal{A}(p,\omega)$ is the Bergman space A^p.
Moreover the following asymptotic relation holds

$$||f||_{\mathcal{A}(p,\omega)} \asymp \sum_{j=0}^{n-1} |f^{(j)}(0)| + \Big(\int_{\mathbf{D}} |f^{(n)}(z)|^p (1-|z|^2)^{pn} \frac{\omega(|z|)}{1-|z|} dA(z) \Big)^{1/p}, \quad (1)$$

where the notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$. (see, for example [4]). Also, it is well known that the point evaluations are bounded linear functionals on $\mathcal{A}(p,\omega)$ and for every $f \in \mathcal{A}(p,\omega)$, the following estimate holds

$$|f^{(n)}(z)| \le C \frac{||f||_{\mathcal{A}(p,\omega)}}{\omega(|z|)(1-|z|^2)^{1/p+n}}; \quad z \in \mathbf{D}. \quad (2)$$

Now we define the Bloch-type spaces of holomorphic functions. The Bloch-type spaces $\mathcal{B}_\nu(\mathbf{D}) = \mathcal{B}_\nu$ consists of all $f \in H(\mathbf{D})$ such that

$$||f||_{\mathcal{B}_\nu} := |f(0)| + b_\nu(f) = |f(0)| + \sup_{z \in \mathbf{D}} \nu(z)|f'(z)| < \infty, \quad (3)$$

where ν is a positive continuous radial weight on $\mathbf{D}$ such that $\nu(|z|)$ decreasingly converges to 0 as $|z| \to 1$. The little Bloch-type space $\mathcal{B}_{\nu,0}(\mathbf{D}) = \mathcal{B}_{\nu,0}$ consists of all $f \in H(\mathbf{D})$ such that

$$\lim_{|z|\to 1} \nu(z)|f'(z)| = 0.$$

With the norm $||\cdot||_{\mathcal{B}_\nu}$ the Bloch-type space $\mathcal{B}_\nu$ is a Banach space and the little Bloch-type space $\mathcal{B}_{\nu,0}$ is a closed subspace of the Bloch-type space.

Let g, $h \in H(\mathbf{D})$ and φ be a holomorphic self-map of $\mathbf{D}$. For a non-negative integer n, we define a linear operator $I^n_{h,\varphi}$ as

$$I^n_{h,\varphi} f(z) = \int_0^z f^{(n)}(\varphi(\zeta))\, h(\zeta)\, d\zeta, \quad f \in H(\mathbf{D}).$$

The operator $I^n_{h,\varphi}$ induces many known operators. When $\varphi(z) = z$, we drop φ and simply write I^n_h for $I^n_{h,\varphi}$. If $n = 0$ and $h(z) = g'(z)$, then we get the operator $T_{g,\varphi}$ induced by g and φ as

$$T_{g,\varphi} f(z) = \int_0^z f(\varphi(\zeta)) dg(\zeta) = \int_0^z f(\varphi(\zeta)) g'(\zeta) d\zeta = \int_0^1 f(\varphi(tz))\, z\, g'(tz) dt.$$

The operator $T_{g,\varphi}$ can be viewed as a generalization of the Riemann-Stieltjes operator T_g induced by g, defined by

$$T_g f(z) = \int_0^z f(\zeta)dg(\zeta) = \int_0^1 f(tz)zg'(tz)dt, \quad z \in \mathbf{D}.$$

If $n = 1$, $h(z) = g(z)$ and $\varphi(z) = z$, then we get the operator J_g, defined by Yoneda in [22] as

$$J_g f(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta, \quad z \in \mathbf{D}.$$

For more about operators of the type $I^n_{h,\varphi}$, we refer [1]-[22].
Throughout this paper constants are denoted by C, they are positive and not necessarily the same at each occurrence.

2 Main Results

In this section, we characterize boundedness and compactness of $I^n_{h,\varphi}$ weighted Bergman spaces to Bloch-type spaces of holomorphic functions.

Theorem 2.1 *Let $0 < p < \infty$, ν a radial weight, ω a normal weight, $h \in H(\mathbf{D})$ and φ be a holomorphic self-map on $\mathbf{D}$. Then $I^n_{h,\varphi} : \mathcal{A}(p,\omega) \to \mathcal{B}_\nu$ is bounded if and only if*

$$M := \sup_{z\in\mathbf{D}} \frac{\nu(z)|h(z)|}{\omega(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{p}+n}} < \infty. \tag{4}$$

Moreover

$$||I^n_{h,\varphi}||_{\mathcal{A}(p,\omega)\to\mathcal{B}_\nu} \asymp M. \tag{5}$$

Proof. Suppose that (4) holds. By (2) for $f \in \mathcal{A}(p,\omega)$ we have

$$\nu(z)|(I^{(n)}_{h,\varphi} f)'(z)| = \nu(z)|f^{(n)}(z)||h(z)| \le C\frac{\nu(z)|h(z)|}{\omega(|z|)(1-|\varphi(z)|^2)^{\frac{1}{p}+n}}||f||_{\mathcal{A}(p,\omega)}.$$

Since $|(I^n_{h,\varphi} f)'(0)| = 0$, so by (3) we have $I^n_{h,\varphi} : \mathcal{A}(p,\omega) \to \mathcal{B}_\nu$ is bounded and

$$||I^n_{h,\varphi}||_{\mathcal{A}(p,\omega)\to\mathcal{B}_\nu} \le CM. \tag{6}$$

Conversely, suppose that $I^n_{h,\varphi} : \mathcal{A}(p,\omega) \to \mathcal{B}_\nu$ is bounded. For $z \in \mathbf{D}$, consider the function

$$f_z(\zeta) = \frac{(1-|\varphi(z)|^2)^{t+1}}{\omega(|\varphi(z)|)(1-\overline{\varphi(z)}\zeta)^{\frac{1}{p}+t+1}}.$$

It is easy to see that $f_z \in \mathcal{A}(p,\omega)$ and $||f_z||_{\mathcal{A}(p,\omega)} \le C$. Thus by the boundedness of $I^n_{h,\varphi} : \mathcal{A}(p,\omega) \to \mathcal{B}_\nu$, we have

$$\begin{aligned}\nu(z)|h(z)||f_z^{(n)}(\varphi(z))| &\le ||I_{h,\varphi}^n f_z||_{\mathcal{B}_\nu} \\ &\le ||I_{h,\varphi}^n||_{\mathcal{A}(p,\omega)\to\mathcal{B}_\nu}||f_z||_{\mathcal{A}(p,\omega)} \\ &\le C||I_{h,\varphi}^n||_{\mathcal{A}(p,\omega)\to\mathcal{B}_\nu}.\end{aligned}$$

Therefore,

$$\frac{\nu(z)|h(z)|}{\omega(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{p}+n}} \le C||I_{h,\varphi}^{(n)}||_{\mathcal{A}(p,\omega)\to\mathcal{B}_\nu}.$$

Taking supremum over $z \in \mathbf{D}$, we have (4). Moreover

$$M \le C||I_{h,\varphi}^{(n)}||_{\mathcal{A}(p,\omega)\to\mathcal{B}_\nu}. \tag{7}$$

Also from (6) and (7), $||I_{h,\varphi}^{(n)}||_{\mathcal{A}(p,\omega)\to\mathcal{B}_\nu} \asymp M$.

The next lemma can be proved in a standard way (see [2], Theorem 3.11).

Lemma 1 *Let $0 < p < \infty$, ν a radial weight, ω a normal weight, $h \in H(\mathbf{D})$ and φ be a holomorphic self-map on $\mathbf{D}$. Then the operator $I_{h,\varphi}^n : \mathcal{A}(p,\omega) \to \mathcal{B}_\nu$ is compact if and only if for any sequence $\{f_j\}$ in $\mathcal{A}(p,\omega)$ which converges to zero uniformly on compact subsets of $\mathbf{D}$, $\{I_{h,\varphi}^n f_j\}$ converges to zero in $\mathcal{B}_\nu$.*

Theorem 2.2 *Let $0 < p < \infty$, ν a radial weight, ω a normal weight, $h \in H(\mathbf{D})$ and φ be a holomorphic self-map on $\mathbf{D}$. Then the operator $I_{h,\varphi}^n : \mathcal{A}(p,\omega) \to \mathcal{B}_\nu$ is compact if and only if*

$$\lim_{r\to 1^-} \sup_{|\varphi(z)|>r} \frac{\nu(z)|h(z)|}{\omega(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{p}+n}} = 0. \tag{8}$$

Proof. Suppose that (8) holds. Let $\{f_j\}$ be a bounded sequence in $\mathcal{A}(p,\omega)$ that converges to zero uniformly on compact subsets of $\mathbf{D}$. Let $M = \sup_j ||f_j||_{\mathcal{A}(p,\omega)} < \infty$. Given $\epsilon > 0$, there exists an $r \in (0,1)$ such that if $|\varphi(z)| > r$, then

$$\frac{\nu(z)|h(z)|}{\omega(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{p}+n}} < \varepsilon.$$

By (2), we have

$$|f_j^n(z)| \le C\frac{||f_j||_{\mathcal{A}(p,\omega)}}{\omega(|z|)(1-|z|^2)^{\frac{1}{p}+n}}.$$

Thus for $z \in \mathbf{D}$ such that $|\varphi(z)| > r$, we have

$$\begin{aligned}\nu(z)|(I_{h,\varphi}^n f_j)'(z)| &= \nu(z)|h(z)||f_j^{(n)}(\varphi(z))| \\ &\le \frac{\nu(z)|h(z)|}{\omega(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{p}+n}}||f_j||_{\mathcal{A}(p,\omega)} \le \epsilon M,\end{aligned}$$

for all j. On the other hand since $f_j \to 0$ uniformly on compact subsets of $\mathbf{D}$, so $|f_j(\varphi(0))| < \epsilon$. Moreover, there exists j_0 such that if $|\varphi(z)| \leq r$ and $j \geq j_0$, then $|f_j^{(n)}(\varphi(z))| < \epsilon$. By taking $f(z) = z^n/n!$ in $\mathcal{A}(p,\omega)$, the boundedness of $I^n_{h,\varphi} : \mathcal{A}(p,\omega) \to \mathcal{B}_\nu$ implies that $N = \sup_{z\in\mathbf{D}} \nu(z)|h(z)| < \infty$. Thus by (**??**), we have

$$
\begin{aligned}
||I^n_{h,\varphi} f_j||_{\mathcal{B}_\nu} &= |f_j(\varphi(0))| + \sup\nolimits_{z\in\mathbf{D}} \nu(z)|(I^n_{h,\varphi} f_j)'(z)| \\
&\leq \varepsilon + \sup_{|\varphi(z)|\leq r} \nu(z)||h(z)|f_j^{(n)}(\varphi(z))| + \sup_{|\varphi(z)|>r} \nu(z)||h(z)|f_j^{(n)}(\varphi(z))| \\
&\leq \varepsilon + \sup_{|\varphi(z)|\leq r} \nu(z)||h(z)|f_j^{(n)}(\varphi(z))| + \varepsilon M < \varepsilon C.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, $I^n_{h,\varphi} : \mathcal{A}(p,\omega) \to \mathcal{B}_\nu$ is compact. Conversely, suppose that $I^n_{h,\varphi} : \mathcal{A}(p,\omega) \to \mathcal{B}_\nu$ is compact and (8) does not holds. Then there exists a positive number δ and a sequence $\{z_j\}$ in $\mathbf{D}$ such that $|\varphi(z_j)| \to 1$ and

$$
\frac{\nu(z_j)|h(z_j)|}{\omega(|\varphi(z_j)|)(1-|\varphi(z_j)|^2)^{\frac{1}{p}+n}} \geq \delta
$$

for all j. For each j, let $a_j = \varphi(z_j)$ and consider the function f_j as

$$
f_j(z) = \frac{(1-|a_j|^2)^{t+1}}{\omega(|a_j|)(1-\overline{a_j}z)^{\frac{1}{p}+t+1}}, \quad z \in \mathbf{D}.
$$

Then f_j is norm bounded and $f_j \to 0$ uniformly on compact subsets of $\mathbf{D}$. It follows that a subsequence of $\{I^n_{h,\varphi} f_j\} \to 0$ in $\mathcal{B}_\nu$. On the other hand

$$
\begin{aligned}
||I^n_{h,\varphi} f_j||_{\mathcal{B}_\nu} &\geq \nu(z_j)|(I^n_{h,\varphi} f_j)'(z_j)| = \nu(z_j)|h(z_j) f_j^{(n)}(\varphi(z_j))| \\
&= \frac{\nu(z_j)|h(z_j)|}{\omega(|a_j|)(1-|\varphi(z_j)|^2)^{\frac{1}{p}+n}} \geq \delta,
\end{aligned}
$$

which is absurd. Hence we are done.

Theorem 2.3 *Let $0 < p < \infty$, ν a radial weight, ω a normal weight, $h \in H(\mathbf{D})$ and φ be a holomorphic self-map on $\mathbf{D}$. Then the operator $I^n_{h,\varphi} : \mathcal{A}(p,\omega) \to \mathcal{B}_{\nu,0}$ is bounded if and only if*

1. $\sup_{z\in\mathbf{D}} \frac{\nu(z)|h(z)|}{\omega(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{p}+n}} < \infty$ *and*

2. $h \in \mathcal{B}_{\nu,0}$.

Proof. Suppose that $I^n_{h,\varphi} : \mathcal{A}(p,\omega) \to \mathcal{B}_{\nu,0}$ is bounded. Then (1) can be proved exactly in the same way as in the proof of Theorem 1. By taking $f(z) = z^n/n!$ in $\mathcal{A}(p,\omega)$ we get $h \in \mathcal{B}_{\nu,0}$.
Conversely, suppose that (1) and (2) are satisfied. Then for each polynomial $p(z)$, we have

$$\nu(z)|(I^n_{h,\varphi}p)'(z)| \leq \nu(z)|h(z)||p^{(n)}(\varphi(z))|$$

from which it follows that $I^n_{h,\varphi}p \in \mathcal{B}_{\nu,0}$. Since the set of all polynomials is dense in $\mathcal{A}(p,\omega)$, we have that for every $f \in \mathcal{A}(p,\omega)$ there is a sequence of polynomials $\{p_m\}$ such that $||f - p_m||_{\mathcal{A}(p,\omega)} \to 0$ as $n \to \infty$. Since the operator $I^n_{h,\varphi} : \mathcal{A}(p,\omega) \to \mathcal{B}_\nu$ is bounded, we have

$$||I^n_{h,\varphi}f - I^n_{h,\varphi}p_m||_{\mathcal{B}_\nu} \leq ||I^n_{h,\varphi}||_{\mathcal{A}(p,\omega)\to\mathcal{B}_\nu}||f - p_m||_{\mathcal{B}_\nu} \to 0$$

as $n \to \infty$. Since $\mathcal{B}_{\nu,0}$ is a closed subspace of $\mathcal{B}_\nu$, we have $I^n_{h,\varphi}(\mathcal{A}(p,\omega)) \subset \mathcal{B}_{\nu,0}$. Therefore, $I^n_{h,\varphi} : \mathcal{A}(p,\omega) \to \mathcal{B}_{\nu,0}$ is bounded.

The following characterization can be proved on similar lines as Lemma 5.2 in [8].

Lemma 2 *A closed set K in $\mathcal{B}_{\nu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \nu(z)|f'(z)| = 0.$$

Theorem 2.4 *Let $0 < p < \infty$, ν a radial weight, ω a normal weight, $h \in H(\mathbf{D})$ and φ be a holomorphic self-map on $\mathbf{D}$. Then the operator $I^n_{h,\varphi}$: $\mathcal{A}(p,\omega) \to \mathcal{B}_{\nu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{\nu(z)|h(z)|}{w(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{1}{p}+n}} = 0. \tag{9}$$

Proof. By Lemma 2 the set $\{I^n_{h,\varphi}f : f \in \mathcal{A}(p,\omega), ||f||_{\mathcal{A}(p,\omega)} \leq 1\}$ has compact closure in $\mathcal{B}_{\nu,0}$ if and only if

$$\lim_{|z|\to 1} \sup\{\nu(z)|(I^n_{h,\varphi}f)'(z)| : f \in \mathcal{A}(p,\omega), ||f||_{\mathcal{A}(p,\omega)} \leq 1\} = 0.$$

Suppose that $f \in \mathcal{A}(p,\omega)$ is such that $||f||_{\mathcal{A}(p,\omega)} \leq 1$ and (9) holds. Then
$\nu(z)|(I^n_{h,\varphi}f)'(z)| = \nu(z)|h(z)f^n(\varphi(z))|$

$$\leq C\frac{\nu(z)|h(z)|}{\omega(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{1}{p}+n}}.$$

Thus
$\sup\{\nu(z)|(I^n_{h,\varphi}f)'(z)| : f \in \mathcal{A}(p,\omega), ||f||_{\mathcal{A}(p,\omega)} \leq 1\}$

$$\leq C\frac{\nu(z)|h(z)|}{\omega(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{1}{p}+n}}$$

and it follows that

$$\lim_{|z|\to 1^-} \sup\{\nu(z)|(I^n_{h,\varphi}f)'(z)| : f \in \mathcal{A}(p,\omega), ||f||_{\mathcal{A}(p,\omega)} \leq 1\} = 0.$$

Hence $I^n_{h,\varphi} : \mathcal{A}(p,\omega) \to \mathcal{B}_{\nu,0}$ is compact.
Conversely, suppose that $I^n_{h,\varphi} : \mathcal{A}(p,\omega) \to \mathcal{B}_{\nu,0}$ is compact. Using the same test as in the proof of Theorem 2, we have

$$\lim_{|\varphi(z)|\to 1^-} \frac{\nu(z)|h(z)|}{\omega(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{1}{p}+n}} = 0. \tag{10}$$

Since $I^n_{h,\varphi} : \mathcal{A}(p,\omega) \to \mathcal{B}_{\nu,0}$ is bounded. By Theorem 3, $h \in \mathcal{B}_{\nu,0}$. It is easy to show that $h \in \mathcal{B}_{\nu,0}$ and (11) is equivalent to (10).

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