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Products of simple group involving the alternating A_8

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Abstract

In this note, we will find the structure of the finite simple groups G with two subgroups A and B such that $G = AB$, where A is a non-abelian simple group and B is isomorphic to the alternating group on eight letters.

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1 Introduction

Let G be a group with subgroups A and B. If $G = AB$, then G is called a factorizable group and $G = AB$ is called a factorization of G. Sometimes we say that G is a product of two subgroups A and B. It is an interesting problem to know the groups with proper factorization. Of course not every group has a proper factorization, for example an infinite group with all proper subgroups finite has no proper factorization, $L_2(13)$ and also the Janko simple group J_1 of order 175560 have no proper factorization.

A factorization $G = AB$ is called maximal if both factors A and B are maximal subgroups of G. In [1], all the maximal factorizations of all the finite simple groups and their automorphism groups are found. In [2], all the factorizations of the alternating and symmetric groups are found with both factors simple.

Here we quote some results concerning the alternating groups in a factorization. In [3], factorizable groups where one factor is a non-abelian simple group and the other factor is isomorphic to the alternating group on 5 letters are classified. Also in [4], the structure of a finite factorizable group with one factor a simple group and the other factor isomorphic to the symmetric group on 6 letters is determined. In [5], the structure of factorizable groups $G = AB$

where $A \cong A_7$ and $B \cong S_n$ was given. In [6], the structure of the finite simple factorizable groups $G = AB$ such that A is a non-abelian simple group and $B \cong A_7$, the symmetric group on seven letters is classified. As a development of the topics, we determined the structure of products of a non-abelian simple group with an alternating group of degree eight.

2 Preliminary results

In this section we obtain results which are needed in the proof of our main theorem. Suppose Ω is a set of cardinality m and G is a k-homogeneous, $1 \leq k \leq m$, group on Ω. If H is a k-homogeneous subgroup of G, then it is easy to get that the orders of subgroups of alternating group A_8 are: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 16, 18, 20, 21, 24, 30, 32, 36, 48, 56, 60, 64, 72, 96, 120, 144, 168, 180, 192, 288, 360, 576, 720, 1344, 2520, 20160. Thus the indexes of subgroups of A_8 are: 1, 8, 15, 28, 35, 56, 70, 105, 112, 120, 140, 168, 210, 280, 315, 336, 360, 420, 560, 630, 672, 840, 960, 1008, 1120, 1260, 1344, 1680, 2016, 2240, 2520, 2880, 3360, 4032, 5040, 6720, 10080, 20160. Therefore A_8 has transitive action on sets of cardinality equal to any of the latter numbers. It is well-known that $A_8 \cong L_4(2)$ has a 2-transitive action on 15 points [7]. Since we need factorizations of the alternating group involving A_8, hence using [1], we will prove the following results.

Lemma 2.1 *Let A_m denote the alternating group of degree m. If $A_m = AB$ is a non-trivial factorization of A_m with A a non-abelian simple group of A_m and $B \cong A_8$, then one of the following cases occurs:*

(a) $A_m = A_{m-1}A_8$, where m =15, 28, 35, 56, 70, 105, 112, 120, 140, 168, 210, 280, 315, 336, 360, 420, 560, 630, 672, 840, 960, 1008, 1120, 1260, 1344, 1680, 2016, 2240, 2520, 2880, 3360, 4032, 5040, 6720, 10080, 20160.

(b) $A_{15} = A_{13}A_8$.

(c) $A_{12} = M_{12}A_8$

Proof. It is obvious that m is at least 9. By Theorem D of [1], we have that either $m = 6, 9$ or 10 or one of A or B is k-homogeneous on m letters. Since $m = 6, 8 or 10$, A_m does not involve A_8 and so we consider the following cases.

Case (i): $A_{m-k} \trianglelefteq A \leq S_{m-k} \times S_k$ for some k with $1 \leq k \leq 5$, and B is k-homogeneous on m letters.

Since A is assumed to be simple we obtain $A_{m-k} = 1$ or A. If $A_{m-k} = 1$, then $m - k = 1$ or 2, hence $k = m - 1$ or $m - 2$. But then from $1 \leq k \leq 5$ we obtain $2 \leq m \leq 6$ or $3 \leq m \leq 7$, a contradiction because $m \geq 9$. Therefore $A = A_{m-k}$ and $B \cong A_8$ is k-homogeneous on m letters, $1 \leq k \leq 5$. If $k = 1$, then the size of the set Ω on which A_8 can act transitively is as statd in the Lemma and all the factorizations in case (a) occur. If $k \geq 2$, then $m = 8$ or 15. If $m = 15$, then $A_8 \cong L_4(2)$, where $L_4(2)$ is the projective special linear group of degree 4 over field of order 2, has a transitive action on 15 letters and hence $A_{15} = A_{14}A_8$ and $A_{15} = A_{13}A_8$ which is case (b).

Case (ii): $A_{m-k} \trianglelefteq B \leq S_{m-k} \times S_k$ for some k with $1 \leq k \leq 5$, and A is k-homogeneous on m letters.

Since $B \cong A_8$ we obtain $A_{m-k} = 1$ or B and so $m - k = 1, 2$ or 8. From $1 \leq k \leq 5$, we have $2 \leq m \leq 6$, $3 \leq m \leq 9$ or $9 \leq m \leq 13$. Therefore, we know that only $m = 9, 10, 11, 12$ or 13 are possible which is correspond to $k = 1, 2, 3, 4, 5$ respectively. But now from Theorem 4.11 and page 197 of [7], and [8], for possible (m, k) we have: (m, k)=(12,4), $A_{12} = M_{12}A_8$, and this is the possibility in case (c) of the Lemma. □

3 Main Results

These are the main results of the paper. To find the structure of the factorizable simple groups $G = AB$ with A simple and $B \cong A_8$, we need to know about the primitive groups of certain degrees which are equal to the indices of subgroups in A_8. Simple primitive groups of degree at most 2500 and 4096 are given in [10] and [11] respectively, and the index of most of the subgroups of A_8 are less than 4096 except the four indices which are 5040, 6720, 10080, and 20160.

Lemma 3.1 *Let G be a non-abelian simple group which is not an alternating group. If G is a primitive group of degree 5040, 6720, 10080, or 20160, then G does not have a factorization $G = A_nB$ with A simple and $B \cong A_8$.*

Proof. From Classification Theorem for the finite simple groups, G is isomorphic either a sporadic simple group or a simple group of Lie type. From [12], there is no factorization as mentioned in the Lemma for a sporadic group. Thus we assume that G is a simple group of Lie type. If the rank of G is 1 or 2, then from [13], here does not have the desired factorization. Hence we we consider that the Lie rank of G is at least 3. We consider the minimum index of a subgroup of a simple group of Lie type (see [14]).

Case (a). $G = L_n(q), n \geq 4$.

In this case, the minimum index of a proper subgroup of G is $\frac{q^n-1}{q-1}$. If $\frac{q^n-1}{q-1} \leq$ 20160, then we have the following possibilities for G: $L_4(q)$, $q = 2, 3, \cdots, 25$, $L_5(q), q = 2, 3, \cdots, 9, 11$, $L_6(2)$, $L_6(3)$, $L_6(4)$, $L_6(5)$, $L_6(7)$, $L_7(2)$, $L_7(3)$, $L_7(4)$, $L_7(5)$, $L_8(2)$, $L_8(3)$, $L_9(2)$, $L_9(3)$, $L_{10}(2)$, $L_{11}(2)$, $L_{12}(2)$, $L_{13}(2)$, or $L_{14}(2)$.

$L_4(2) \cong A_8$ is not the case. If $L_4(3) = A_8A_n$, then since $13 \big| |L_4(3)|$, $n \geq 13$ and so $11 \big| |L_4(3)|$, a contradiction. Similarly we can rule out the above possibilities except $L_6(2)$ and $L_6(3)$. If $L_6(2) = A_8A_n$, then since $|L_6(2)| = 2 \cdot 3^4 \cdot 5^2 \cdot 7^2$, $n = 7, 8, 9, 10$. If $n = 7$ or 8, then $L_6(2) = A_8$, a contradiction. If $n = 8, 9$, or 10, then $8 \big| |L_6(2)|$, a contradiction. Thus $L_6(3) = A_8A_n$. Since $13 \big| |L_6(3)|$, then $n = 13, 14, 15$, or 16. We can rule out the case as the proof of $L_6(2)$.

Case (b). $G = U_n(q)$, $n \geq 6$.

In this case, the proper subgroups have index at least

$$\frac{(q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})}{q^2 - 1}.$$

If

$$\frac{(q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})}{q^2 - 1} \leq 20160,$$

then we have the following possibilities for G: $U_6(2)$, $U_7(2)$ or $U_8(2)$. From [19], $U_6(2)$ and $U_7(2)$ have no maximal subgroup of index 5040, 6720, 10080, or 20160. Therefore, $U_8(2) = A_8A_n$. Since $17 \big| |U_8(2)|$, then $n \geq 17$ and so $13 \big| |U_8(2)|$, a contradiction.

Case (c). $G = S_{2m}(q)$, $m \geq 3$.

In this case if $q > 2$ then the index of a proper subgroup of G is at least $\frac{q^{2m}-1}{q-1}$ and if q=2 then this number is $2^{m-1}(2^m - 1)$.

If $q = 2$, then $2^m(2^m - 1) \leq 20160$, and so we have the following possibilities for G: $S_6(2), S_8(2), S_{10}(2), S_{12}(2)$, or $S_{14}(2)$. If $q > 2$, then $\frac{q^{2m}-1}{q-1} \leq 20160$, and so the possibilities for G are: $S_6(3), S_6(4), S_6(5), S_6(7)$, or $S_8(3)$. From [15], it is easy to get that the groups $S_6(2), S_6(3)$ or $S_8(2)$ have no maximal subgroup of index 5040, 6720, 10080, or 20160. For the groups $S_{10}(2)$, $S_{12}(2)$, $S_{14}(2)$, $S_6(4)$, $S_6(5)$, $S_6(7)$ and $S_8(3)$ similar arguments as used as case (b) rule out the possibility of factorizing these groups as product of a simple group and a group isomorphic to A_8.

Case (d). $G = O^{\varepsilon}_{2m}(q)$, $m \geq 4$, $\varepsilon = \pm$.

In this case, the proper subgroups have index at least $\frac{(q^m-1)(q^{m-1}+1)}{q-1}$ where $\varepsilon = +$ and at least $\frac{(q^m+1)(q^{m-1}-1)}{q-1}$ when $\varepsilon = -$ except for the case $(q, \varepsilon) = (2, +)$ when a proper subgroup has index at least $2^{m-1}(2^m - 1)$. Then we have the following possibilities for G of degree less than or equal to 20160: $O^{\pm}_8(2)$, $O^{\pm}_{10}(2)$,$O^{\pm}_{12}(2)$, $O^{\pm}_{14}(2)$, $O^{\pm}_8(3)$, $O^{\pm}_{10}(3)$, $O^{\pm}_8(4)$, or $O^{\pm}_{10}(4)$. From [15], $O^{\pm}_8(2)$, $O^{\pm}_{10}(2)$ and $O^{\pm}_8(3)$ have no maximal subgroup of index 5040, 6720, 10080, or 20160. For the remaining cases, using order we can rule out these possibilities.

Case (e). $G = O_{2m+1}(q)$, $m \geq 3$, q odd.

In this case, the proper subgroups have index at least $\frac{q^{2m}-1}{q-1}$ except when $q = 3$ and in the latter case $\frac{q^{2m}-q^m}{2}$. Then we have the following possibilities for G of degree less than or equal to 20160: $O_7(3)$, $O_7(5)$, $O_7(7)$, or $O_9(3)$. We can exclude the possibilities by using order of G.

Case (f). G is an exceptional simple groups of Lie type.

From [1], we can find that there is no group involving A_8.

The Lemma is proved. □

Table 1. Non-abelian simple primitive groups of degree n

degree	group
15	$A_{15},\ A_8$
28	$A_{28}, L_2(8)$
35	A_{35}, A_8
56	$A_{56}, L_3(8)$
70	A_{70}
105	A_{105}
112	$A_{112}, U_4(3)$
120	$A_{120}, A_9, L_2(16), L_3(8), S_4(4), S_6(2)$
140	A_{140}
168	A_{168}
210	A_{210}
280	$A_{280}, A_9, L_3(4), U_4(3), J_2$
315	$A_{315}, S_6(2), J_2$
336	$A_{336}, S_6(2)$
360	A_{360}
420	A_{420}
560	$A_{560},\ {}^2B_2(8)$
630	A_{630}
672	$A_{672}, U_6(2), M_{22}$
840	A_{840}, A_9, J_2
960	$A_{960}, S_6(2), O_8^+(2)$
1008	A_{1008}, J_2
1120	$A_{1120}, S_6(3), O_7(3)$
1260	A_{1260}
1344	A_{1344}
1680	A_{1680}
2016	$A_{2016}, L_2(64), S_4(8), S_6(4)$
2240	A_{2240}
2520	$A_{2520}, A_{10}, A_{11}, A_{12}$
2880	A_{2880}
3360	A_{3360}
4032	A_{4032}

Theorem 3.2 *Let $G = AB$ is a non-trivial factorization of a simple group G with A a non-abelian simple group and $B \cong A_8$, then one of the following cases occurs:*

(a) $A_m = A_{m-1}A_8$, *where m=15, 28, 35, 56, 70, 105, 112, 120, 140, 168, 210, 280, 315, 336, 360, 420, 560, 630, 672, 840, 960, 1008, 1120, 1260, 1344, 1680, 2016, 2240, 2520, 2880, 3360, 4032, 5040, 6720, 10080, 20160.*

(b) $A_{15} = A_{13}A_8$.

(c) $A_{12} = M_{12}A_8$

(d) $S_6(2) = U_3(3)A_8 = L_2(8)A_8$

Proof. Assume that $G = AB$ is a non-trivial factorization of a simple group G with A a non-abelian simple group and $B \cong A_8$. If M is a maximal subgroup of G containing A, then $G = MB$, hence $|G : M|\,||B : M \cap B|$. Since $d = |B : B \cap M|$ is equal to the index of a subgroup of A_8, therefor G is primitive permutation group of degree d. We know that d=1, 8, 15, 28, 35, 56, 70, 105, 112, 120, 140, 168, 210, 280, 315, 336, 360, 420, 560, 630, 672, 840, 960, 1008, 1120, 1260, 1344, 1680, 2016, 2240, 2520, 2880, 3360, 4032, 5040, 6720, 10080, 20160. It is easy to see that $d \neq 1$ or 8. By Lemma 3.1, if $d =$ 5040, 6720, 10080, or 20160, G is isomorphic to an alternating group of these degrees. If G is an alternating group, then Lemma 2.1 implies that the cases (a) (b) and (c) is as in the Theorem. The following, we will prove if G is not an alternating group, then since the remaining degrees d are less than 4096, we can establish the Table 1 by using [10] and [11].

By [16], [17], [12] and [13], we only consider the following groups: $S_4(4)$, $S_4(8)$, $S_6(2)$, $S_6(4)$, $S_6(3)$, $O_7(3)$, $O_8^+(2)$, ${}^2B_2(8)$. Let M be a maximal subgroup of G containing A.

If $G = S_4(4)$, then $d = |G : M| = 120$. According to [15], we get $M \cong L_2(16) : 2$ and so $A = L_2(16)$, But now order consider rule out the case.

If $G =^2 B_2(8)$, then $d = |G : M| = 560$. By [15], we have $M \cong 13 : 4$, and so $A \leq M$ is a simple group, a contradiction.

If $G = S_6(2)$, then $d = |G : M| = 120, 315, 336$ or 960. If $d = 120$, then $M \cong U_3(3) : 2$ and so $A \cong U_3(3)$. Therefore we have $S_6(2) = U_3(3)A_8$. This factorization is possible. In fact, the intersection the two factors is a group of order 12, which is contained in both A_8 and $U_3(3)$. If $d = 315$ or 336, then from [15], we rule out this case. If $d = 960$, then we have $S_6(2) = L_2(8)A_8$. This is case (d).

If $G = S_6(3)$, then from [15], there is no maximal subgroup of index 2016 and so we rule out the case.

If $G = O_7(3)$, then $d = |G : M| = 1120$ and from [15], we have $M \cong 3^{3+3} : L_3(3)$, and $A = L_3(3)$. Thus $O_7(3) = L_3(3)A_8$. Order consideration rule out the case.

If $G = O_8^+(2)$, then $d = 960$. From [15], we have $A = A_9$. Hence we have $O_8^+(2) = A_8A_9 = A_9$, a contradiction.

If $G = S_4(8)$, then $d = |G : M| = 2016$ and $|A| = 2^6 \cdot 3^2 \cdot 7 \cdot 13 \cdot |A \cap A_8|$. Note that $\pi(A \cap A_8) \subset \{2, 3, 5, 7\}$. From [20] and order consideration, we have $A =^3 D_4(2)$. Then since $A \leq M$, $|G : A| = 5 \geq d = |G : M|$, a contradiction.

This completes the proof of the Theorem. □

4 Conclusion

In this note, we give the structure of product of a simple group with an alternating group A_8. We know that $A_n = n!$, then determining the structure of subgroups of A_n is difficult if $n \geq 9$. So how to determine the structure of product of simple groups with A_n for $n > 8$ is a very interesting problem.

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