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Right and left closure operators

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Abstract

In this paper, we investigate the properties of right and left closure on a generalized residuated lattice. In particular, we study the relations between right (left) closure (interior) operators and residuated , Galois connetions with isotone and antitone maps.

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1 Introduction

Galois connections and closure operators on fuzzy sets introduced by Bělohlávek [1,2] are important mathematical tools [1-6]. Recently, Bělohlávek [1-3] investigate the properties of fuzzy relations and similarities on a residual lattice which supports part of foundation of theoretic computer science. Georgescu and Popescue [4.5] introduced non-commutative fuzzy Galois connection in a generalized residuated lattice which is induced by two implications.

In this paper, we investigate the properties of right and left closure on a generalized residuated lattice. In particular, we study the relations between right (left) closure (interior) operators and residuated , Galois connetions with isotone and antitone maps.

2 Preliminaries

Definition 2.1 [4,5] A structure $(L, \vee, \wedge, \odot, \rightarrow, \Rightarrow, 0, 1)$ is called a *generalized residuated lattice* if it satisfies the following conditions:

(GR1) $(L, \vee, \wedge, 1, 0)$ is a bounded where 1 is the universal upper bound and 0 denotes the universal lower bound;

(GR2) $(L, \odot, 1)$ is a monoid;

(GR3) it satisfies a residuation , i.e.

$$a \odot b \leq c \text{ iff } a \leq b \rightarrow c \text{ iff } b \leq a \Rightarrow c.$$

We call that a generalized residuated lattice has the law of double negation if $a = (a^*)^0 = (a^0)^*$ where $a^0 = a \rightarrow 0$ and $a^* = a \Rightarrow 0$.

Remark 2.2 [4-8] (1) A generalized residuated lattice is a residuated lattice ($\rightarrow = \Rightarrow$) iff $\odot$ is commutative.

(2) A left-continuous t-norm $([0,1], \leq, \odot)$ defined by $a \rightarrow b = \bigvee\{c \mid a \odot c \leq b\}$ is a residuated lattice

(3) Let $(L, \leq, \odot)$ be a quantale. For each $x, y \in L$, we define

$$x \rightarrow y = \bigvee\{z \in L \mid z \odot x \leq y\}, \quad x \Rightarrow y = \bigvee\{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, that is,

$(x \odot y) \leq z$ iff $x \leq (y \rightarrow z)$ iff $y \leq (x \Rightarrow z)$. Hence $(L, \vee, \wedge, \odot, \rightarrow, \Rightarrow, 0, 1)$ is a generalized residuated lattice.

(4) A pseudo MV-algebra is a generalized residuated lattice with the law of double negation.

In this paper, we assume $(L, \wedge, \vee, \odot, \rightarrow, \Rightarrow, 0, 1)$ is a generalized residuated lattice with the law of double negation and if the family supremum or infumum exists, we denote $\bigvee$ and $\bigwedge$.

Lemma 2.3 [4-8] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$ for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(2) $x \odot y \leq x \wedge y \leq x \vee y$.

(3) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$ for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(4) $x \rightarrow (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \rightarrow y_i)$, for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(5) $(\bigwedge_{i\in\Gamma} x_i) \rightarrow y \geq \bigvee_{i\in\Gamma}(x_i \rightarrow y)$, for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(6) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ and $(x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$.

(7) $x \rightarrow (y \Rightarrow z) = y \Rightarrow (x \rightarrow z)$ and $x \Rightarrow (y \rightarrow z) = y \rightarrow (x \Rightarrow z)$.

(8) $x \odot (x \rightarrow y) \leq y$ and $(x \Rightarrow y) \odot x \leq y$.

(9) $(x \Rightarrow y) \odot (y \Rightarrow z) \leq x \Rightarrow z$ and $(y \rightarrow z) \odot (x \rightarrow y) \leq x \rightarrow z$.
(10) $(x \Rightarrow z) \leq (y \odot x) \Rightarrow (y \odot z)$ and $(x \rightarrow z) \leq (x \odot y) \rightarrow (z \odot y)$.
(11) $(x \Rightarrow y) \leq (y \Rightarrow z) \rightarrow (x \Rightarrow z)$ and $(y \Rightarrow z) \leq (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$
(12) $x_i \rightarrow y_i \leq (\bigwedge_{i\in\Gamma} x_i) \rightarrow (\bigwedge_{i\in\Gamma} y_i)$ for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.
(13) $x_i \rightarrow y_i \leq (\bigvee_{i\in\Gamma} x_i) \rightarrow (\bigvee_{i\in\Gamma} y_i)$ for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.
(14) $x \rightarrow y = 1$ iff $x \leq y$.
(15) $x \rightarrow y = y^0 \Rightarrow x^0$ and $x \Rightarrow y = y^* \rightarrow x^*$.
(16) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.
(17) $\bigwedge_{i\in\Gamma} x_i^0 = (\bigvee_{i\in\Gamma} x_i)^0$ and $\bigvee_{i\in\Gamma} x_i^0 = (\bigwedge_{i\in\Gamma} x_i)^0$.

Definition 2.4 Let X be a set. A function $e_X : X \times X \rightarrow L$ is called:
(E1) $e_X(x, x) = 1$ for all $x \in X$,
(R) $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$.
Then e_X is called a right preorder. If e_X satisfies (E1) and
(L) $e_X(y, z) \odot e_X(x, y) \leq e_X(x, z)$, for all $x, y, z \in X$.
Then e_X is called a left preorder.
The pair (X, e_X) is a right preorder (resp. left-preorder) set.

Example 2.5 (1) We define two functions $e_L^{\Uparrow}, e_L^{\uparrow} : L \times L \rightarrow L$ as $e_L^{\Uparrow}(x, y) = x \Rightarrow y$ and $e_L^{\uparrow}(x, y) = x \rightarrow y$. Then $e_L^{\Uparrow}$ is a right preorder and $e_L^{\uparrow}$ is a left preorder.

(2) We define two functions $e_{L^X}^{\Uparrow}, e_{L^X}^{\uparrow} : L^X \times L^X \rightarrow L$ as

$$e_{L^X}^{\Uparrow}(A, B) = \bigwedge_{x\in X} (A(x) \Rightarrow B(x)),\ e_{L^X}^{\uparrow}(A, B) = \bigwedge_{x\in X} (A(x) \rightarrow B(x)).$$

Then $e_{L^X}^{\Uparrow}$ is a right preorder and $e_{L^X}^{\uparrow}$ is a left preorder.

Definition 2.6 [1-4] Let X and Y be two sets. Let $F, H : L^X \rightarrow L^Y$ and $G, K : L^Y \rightarrow L^X$ be operators.

(1) The pair (F, G) is called a *residuated connection* between X and Y if for $A \in L^X$ and $B \in L^Y$, $F(A) \leq B$ iff $A \leq G(B)$.

(2) The pair (H, K) is called a *Galois connection* between X and Y if for $A \in L^X$ and $B \in L^Y$, $B \leq H(A)$ iff $A \leq K(B)$.

3 Right and left closure operators

Definition 3.1 (1) A map $G : L^X \rightarrow L^Y$ is a right isotone map if for all $A, B \in L^X$, $e_{L^X}^{\Uparrow}(A, B) \leq e_{L^Y}^{\Uparrow}(G(A), G(B))$.

(2) A map $G : L^X \rightarrow L^Y$ is a left isotone map if for all $A, B \in L^X$, $e_{L^X}^{\uparrow}(A, B) \leq e_{L^Y}^{\uparrow}(G(A), G(B))$.

(3) A map $G : L^X \to L^Y$ is a right antitone map if for all $A, B \in L^X$, $e^{\uparrow}_{L^X}(A, B) \le e^{\Uparrow}_{L^Y}(G(B), G(A))$.

(4) A map $G : L^X \to L^Y$ is a left antitone map if for all $A, B \in L^X$, $e^{\Uparrow}_{L^X}(A, B) \le e^{\uparrow}_{L^Y}(G(B), G(A))$.

Theorem 3.2 (1) *A map $G : L^X \to L^Y$ is a left isotone map iff $\alpha \odot G(A) \le G(\alpha \odot A)$ and $G(A) \le G(B)$ for $A \le B$ iff $G(\alpha \Rightarrow A) \le \alpha \Rightarrow G(A)$ and $G(A) \le G(B)$ for $A \le B$.*

(2) *A map $G : L^X \to L^Y$ is a right isotone map iff $G(A) \odot \alpha \le G(A \odot \alpha)$ and $G(A) \le G(B)$ for $A \le B$ iff $G(\alpha \to A) \le \alpha \to G(A)$ and $G(A) \le G(B)$ for $A \le B$.*

(3) *A map $G : L^X \to L^Y$ is a right antitone map iff $G(\alpha \odot A) \le \alpha \to G(A)$ and $G(B) \le G(A)$ for $A \le B$ iff $G(A) \odot \alpha \le G(\alpha \Rightarrow A)$ and $G(B) \le G(A)$ for $A \le B$.*

(4) *A map $G : L^X \to L^Y$ is a left antitone map iff $G(A \odot \alpha) \le \alpha \Rightarrow G(A)$ and $G(B) \le G(A)$ for $A \le B$ iff $\alpha \odot G(A) \le G(\alpha \to A)$ and $G(B) \le G(A)$ for $A \le B$.*

(5) *If $G : L^X \to L^Y$ is a left isotone map, then $G^0 : L^X \to L^Y$ is a right antitone map.*

(6) *If $G : L^X \to L^Y$ is a right isotone map, then $G^* : L^X \to L^Y$ is a left antitone map.*

(7) *If $G : L^X \to L^Y$ is a right antitone map, then $G^* : L^X \to L^Y$ is a left isotone map.*

(8) *If $G : L^X \to L^Y$ is a left antitone map, then $G^0 : L^X \to L^Y$ is a right isotone map.*

Proof (1) First, we show that $G : L^X \to L^Y$ is a left isotone map iff $\alpha \odot G(A) \le G(\alpha \odot A)$ and $G(A) \le G(B)$ for $A \le B$. Let $G : L^X \to L^Y$ be a left isotone map. Then $e^{\uparrow}_{L^X}(A, B) \le e^{\uparrow}_{L^Y}(G(A), G(B))$. Put $B = \alpha \odot A$. Then

$$\alpha \le e^{\uparrow}_{L^X}(A, \alpha \odot A) \le e^{\uparrow}_{L^Y}(G(A), G(\alpha \odot A)).$$

Hence $\alpha \odot G(A) \le G(\alpha \odot A)$.

Conversely, put $\alpha = e^{\uparrow}_{L^X}(A, B)$.

$$e^{\uparrow}_{L^X}(A, B) \odot G(A) \le G(e^{\uparrow}_{L^X}(A, B) \odot A) \le G(B).$$

Hence $e^{\uparrow}_{L^X}(A, B) \le e^{\uparrow}_{L^Y}(G(A), G(B))$.

Second, we show that $\alpha \odot G(A) \le G(\alpha \odot A)$ and $G(A) \le G(B)$ for $A \le B$ iff $G(\alpha \Rightarrow A) \le \alpha \Rightarrow G(A)$ and $G(A) \le G(B)$ for $A \le B$.

Since $\alpha \odot G(\alpha \Rightarrow A) \le G(\alpha \odot (\alpha \Rightarrow A)) \le G(A)$, then $G(\alpha \Rightarrow A) \le \alpha \Rightarrow G(A)$.

Conversely, since $G(\alpha \Rightarrow \alpha \odot A) \leq \alpha \Rightarrow G(\alpha \odot A)$ iff $\alpha \odot G(\alpha \Rightarrow \alpha \odot A) \leq G(\alpha \odot A)$, we have

$$\alpha \odot G(A) \leq \alpha \odot G(\alpha \Rightarrow \alpha \odot A) \leq G(\alpha \odot A).$$

(2) First, we show that $G : L^X \to L^Y$ is a right isotone map iff $G(A) \odot \alpha \leq G(A \odot \alpha)$ and $G(A) \leq G(B)$ for $A \leq B$. Let $G : L^X \to L^Y$ be a right isotone map. Then $e^{\Uparrow}_{L^X}(A, B) \leq e^{\Uparrow}_{L^Y}(G(A), G(B))$. Put $B = A \odot \alpha$. Then

$$\alpha \leq e^{\Uparrow}_{L^X}(A, A \odot \alpha) \leq e^{\Uparrow}_{L^Y}(G(A), G(A \odot \alpha)).$$

Hence $G(A) \odot \alpha \leq G(A \odot \alpha)$.

Conversely, put $\alpha = e^{\Uparrow}_{L^X}(A, B)$.

$$G(A) \odot e^{\Uparrow}_{L^X}(A, B) \leq G(A \odot e^{\Uparrow}_{L^X}(A, B)) \leq G(B).$$

Hence $e^{\Uparrow}_{L^X}(A, B) \leq e^{\Uparrow}_{L^Y}(G(A), G(B))$.

Second, we show that $G(A) \odot \alpha \leq G(A \odot \alpha)$ and $G(A) \leq G(B)$ for $A \leq B$ iff $G(\alpha \to A) \leq \alpha \to G(A)$ and $G(A) \leq G(B)$ for $A \leq B$.

Since $G(\alpha \to A) \odot \alpha \leq G((\alpha \to A) \odot \alpha) \leq G(A)$, then $G(\alpha \to A) \leq \alpha \to G(A)$.

Conversely, since $G(\alpha \to A \odot \alpha) \leq \alpha \to G(A \odot \alpha)$ iff $G(\alpha \to A \odot \alpha) \odot \alpha \leq G(A \odot \alpha)$, we have

$$G(A) \odot \alpha \leq G(\alpha \to A \odot \alpha) \odot \alpha \leq G(A \odot \alpha).$$

(3) Let $G : L^X \to L^Y$ be a right antitone map. Then $e^{\uparrow}_{L^X}(A, B) \leq e^{\Uparrow}_{L^Y}(G(B), G(A))$. Put $B = \alpha \odot A$. Then $\alpha \leq e^{\uparrow}_{L^X}(A, \alpha \odot A) \leq e^{\Uparrow}_{L^Y}(G(\alpha \odot A), G(A))$. Then $G(\alpha \odot A) \leq \alpha \to G(A)$.

Conversely, since $G(e^{\uparrow}_{L^X}(A, B) \odot A) \leq e^{\uparrow}_{L^X}(A, B) \to G(A)$ and $G(e^{\uparrow}_{L^X}(A, B) \odot A) \geq G(B)$ for $e^{\uparrow}_{L^X}(A, B) \odot A \leq B$, we have

$$e^{\uparrow}_{L^X}(A, B) \leq G(e^{\uparrow}_{L^X}(A, B) \odot A) \Rightarrow G(A) \leq G(B) \Rightarrow G(A)$$

Second, we show that $G(\alpha \odot A) \leq \alpha \to G(A)$ and $G(B) \leq G(A)$ for $A \leq B$ iff $G(A) \odot \alpha \leq G(\alpha \Rightarrow A)$ and $G(B) \leq G(A)$ for $A \leq B$.

Let $G(\alpha \odot A) \leq \alpha \to G(A)$ and $G(B) \leq G(A)$ for $A \leq B$. Then $G(\alpha \odot A) \odot \alpha \leq G(A)$. Thus

$$G(A) \odot \alpha \leq G(\alpha \odot (\alpha \Rightarrow A)) \odot \alpha \leq G(\alpha \Rightarrow A).$$

Let $G(A) \odot \alpha \leq G(\alpha \Rightarrow A)$ and $G(B) \leq G(A)$ for $A \leq B$. Then $G(A) \leq \alpha \to G(\alpha \Rightarrow A)$. Put $A = \alpha \odot B$. $G(\alpha \odot B) \leq \alpha \to G(\alpha \Rightarrow \alpha \odot B) \leq \alpha \to G(B)$.

(4) It is proved a similar method as in (3).

(5) Let $G : L^X \to L^Y$ be a left isotone map. By Lemma 2.3 (15), $e^{\uparrow}_{L^X}(A,B) \le e^{\uparrow}_{L^Y}(G(A),G(B)) = e^{\Uparrow}_{L^Y}(G^0(B),G^0(A))$.. Hence $G^0 : L^X \to L^Y$ is a right antitone map.

(6) Let $G : L^X \to L^Y$ be a right isotone map. By Lemma 2.3 (15), $e^{\Uparrow}_{L^X}(A,B) \le e^{\Uparrow}_{L^Y}(G(A),G(B)) = e^{\uparrow}_{L^Y}(G^*(B),G^*(A))$.. Hence $G^* : L^X \to L^Y$ is a left antitone map.

(4), (7) and (8) are proved similar methods as in (3), (5) and (6), respectively.

Definition 3.3 A map $C : L^X \to L^X$ is called a right (resp. left) closure operator if it satisfies the following conditions:

(C1) $A \le C(A)$, for all $A \in L^X$.

(C2) $C(C(A)) = C(A)$, for all $A \in L^X$.

(C3) C is a right (resp. left) isotone map.

A map $I : L^X \to L^X$ is called a right (resp. left) interior operator if it satisfies the following conditions:

(I1) $I(A) \le A$, for all $A \in L^X$.

(I2) $I(I(A)) = I(A)$, for all $A \in L^X$.

(I3) I is a right (resp. left) isotone map.

Theorem 3.4 (1) *Let $C : L^X \to L^X$ be a right closure operator. Define a map $I : L^X \to L^X$ as $I(A) = C(A^0)^*$. Then I is a left interior operator.*

(2) *Let $C : L^X \to L^X$ be a left closure operator. Define a map $I : L^X \to L^X$ as $I(A) = C(A^*)^0$. Then I is a right interior operator.*

Proof. (1)

$$\begin{aligned} e^{\uparrow}_{L^X}(A,B) &= e^{\Uparrow}_{L^X}(B^0,A^0) \le e^{\Uparrow}_{L^X}(C(B^0),C(A^0)) \\ &= e^{\uparrow}_{L^X}(C(A^0)^*,C(B^0)^*) = e^{\uparrow}_{L^X}(I(A),I(B)). \end{aligned}$$

(2) It is proved by a similar method as in (1).

Theorem 3.5 *Let $G : L^X \to L^Y$ and $H : L^Y \to L^X$ be two maps.*

(1) *A pair (G,H) is a residuated connection with two right isotone maps G and H iff for all $A \in L^X$ and $B \in L^Y$, $e^{\Uparrow}_{L^Y}(G(A),B) = e^{\Uparrow}_{L^X}(A,H(B))$.*

(2) *A pair (G,H) is a residuated connection with two left isotone maps G and H iff for all $A \in L^X$ and $B \in L^Y$, $e^{\uparrow}_{L^Y}(G(A),B) = e^{\uparrow}_{L^X}(A,H(B))$.*

(3) *A pair (G,H) is a Galois connection with right antitone map G and left antitone map H iff for all $A \in L^X$ and $B \in L^Y$, $e^{\uparrow}_{L^X}(A,H(B)) = e^{\Uparrow}_{L^Y}(B,G(A))$.*

(4) *A pair (G,H) is a Galois connection with left antitone map G and right antitone map H iff for all $A \in L^X$ and $B \in L^Y$, $e^{\Uparrow}_{L^X}(A,H(B)) = e^{\uparrow}_{L^Y}(B,G(A))$.*

Proof. (1) Let (G,H) be a residuated connection. Then $G(A) \leq G(A)$ iff $A \leq H(G(A))$ and $H(B) \leq H(B)$ iff $G(H(B)) \leq B$. Hence $e^{\Uparrow}_{L^Y}(G(A),B) = e^{\Uparrow}_{L^X}(A,H(B))$ from

$$e^{\Uparrow}_{L^Y}(G(A),B) \leq e^{\Uparrow}_{L^X}(H(G(A)),H(B)) \leq e^{\Uparrow}_{L^X}(A,H(B))$$

$$e^{\Uparrow}_{L^X}(A,H(B)) \leq e^{\Uparrow}_{L^Y}(G(A),G(H(B))) \leq e^{\Uparrow}_{L^Y}(G(A),B).$$

Conversely, since $e^{\Uparrow}_{L^Y}(G(A),B) = e^{\Uparrow}_{L^X}(A,H(B))$ for all $A \in L^X$ and $B \in L^Y$, $A \leq H(B)$ iff $G(A) \leq B$. Thus, (G,H) is a residuated connection. Put $B = G(A)$, then $\top = e^{\Uparrow}_{L^Y}(G(A),G(A)) = e^{\Uparrow}_{L^X}(A,H(G(A)))$. So, $A \leq H(G(A))$. Put $A = H(B)$, we similarly obtain $G(H(B)) \leq B$. Thus we obtain two right isotone maps G and H from:

$$e^{\Uparrow}_{L^Y}(G(A),G(B)) = e^{\Uparrow}_{L^X}(A,H(G(B))) \geq e^{\Uparrow}_{L^X}(A,B)$$

$$e^{\Uparrow}_{L^X}(H(A),H(B)) = e^{\Uparrow}_{L^Y}(G(H(A)),B) \geq e^{\Uparrow}_{L^Y}(A,B).$$

(3) Let (G,H) be a Galois connection. Then $G(A) \leq G(A)$ iff $A \leq H(G(A))$ and $H(B) \leq H(B)$ iff $B \leq G(H(B))$. Moreover, since G is a right antitone map and H is a left antitone map, we have

$$e^{\Uparrow}_{L^Y}(B,G(A)) \leq e^{\uparrow}_{L^X}(H(G(A)),H(B)) \leq e^{\uparrow}_{L^X}(A,H(B))$$

$$e^{\uparrow}_{L^X}(A,H(B)) \leq e^{\Uparrow}_{L^Y}(G(H(B)),G(A)) \leq e^{\Uparrow}_{L^Y}(B,G(A)).$$

Hence $e^{\uparrow}_{L^X}(A,H(B)) = e^{\Uparrow}_{L^Y}(B,G(A))$.

Conversely, since $e^{\uparrow}_{L^X}(A,H(B)) = e^{\Uparrow}_{L^Y}(B,G(A))$, $A \leq H(B)$ iff $B \leq G(A)$. Moreover,

$$e^{\Uparrow}_{L^Y}(G(A),G(B)) = e^{\uparrow}_{L^X}(B,H(G(A))) \geq e^{\uparrow}_{L^X}(B,A)$$

$$e^{\uparrow}_{L^X}(H(A),H(B)) = e^{\Uparrow}_{L^Y}(B,G(H(A))) \geq e^{\Uparrow}_{L^Y}(B,A).$$

(2) and (4) are similarly proved as (1) and (3), respectively.

Theorem 3.6 *Let $G : L^X \to L^Y$ and $H : L^Y \to L^X$ be right isotone maps with a residuated connection (G,H). Then the following statements hold:*

(1) *$H \circ G$ is a right closure operator.*

(2) *$G \circ H$ is a right interior operator.*

Proof. (1) Since $A \leq H(G(A))$, $H(G(A)) \leq H(G(H(G(A))))$ for all $A \in L^X$. Since $B \geq G(H(B))$, $G(A) \geq G(H(G(A)))$ and $H(G(A)) \geq H(G(H(G(A))))$. Thus $H(G(A)) = H(G(H(G(A))))$. Since G and H are right isotone maps, $e^{\Uparrow}_{L^X}(A,B) \leq e^{\Uparrow}_{L^X}(H(G(A)),H(G(B)))$.

(2) It is similarly proved as in (1).

Corollary 3.7 *Let $G : L^X \to L^Y$ and $H : L^Y \to L^X$ be left isotone maps with a residuated connection (G, H). Then the following statements hold:*
(1) *$H \circ G$ is a left closure operator.*
(2) *$G \circ H$ is a left interior operator.*

Theorem 3.8 *Let $G : L^X \to L^Y$ be a right antitone map and $H : L^Y \to L^X$ be a left antitone map with a Galois connection (G, H). Then*
(1) *$H \circ G$ is a left closure operator.*
(2) *$G \circ H$ is a right closure operator.*

Proof. (1) Since $A \leq H(G(A))$, then $H(G(A)) \leq H(G(H(G(A))))$ for all $A \in L^X$. Since $B \leq G(H(B))$, then $G(A) \leq G(H(G(A)))$ and $H(G(A)) \geq H(G(H(G(A))))$ because H is a left antitone map. So, $H(G(A)) = H(G(H(G(A))))$.
Since G is a right antitone map and H is a left antitone map, $e^{\uparrow}_{L^X}(A, B) \leq e^{\Uparrow}_{L^X}(G(B), G(A)) \leq e^{\uparrow}_{L^X}(H(G(A)), H(G(B)))$.
(2) It is proved by a similar method as in (1).

Corollary 3.9 *Let $G : L^X \to L^Y$ be a left antitone map and $H : L^Y \to L^X$ be a right antitone map with a Galois connection (G, H). Then*
(1) *$H \circ G$ is a right closure operator.*
(2) *$G \circ H$ is a left closure operator.*

Definition 3.10 For each $A \in L^X$ and $B \in L^Y$ and $R \in L^{X \times Y}$, we define:
(1) $R^{\odot}, {}^{\odot}R : L^X \to L^Y$ is defined as:

$$R^{\odot}(A)(y) = \bigvee_{x \in X} (R(x, y) \odot A(x)), \quad {}^{\odot}R(A)(y) = \bigvee_{x \in X} (A(x) \odot R(x, y)).$$

(2) $R^{\Uparrow}, R^{\uparrow} : L^Y \to L^X$ is defined as:

$$R^{\Uparrow}(B)(x) = \bigwedge_{y \in Y} (R(x, y) \Rightarrow B(y)), \quad R^{\uparrow}(B)(x) = \bigwedge_{y \in Y} (R(x, y) \to B(y)).$$

Theorem 3.11 (1) *$R^{\odot}$ and $R^{\Uparrow}$ are right isotone maps with a residuated connection $(R^{\odot}, R^{\Uparrow})$.*
(2) *$R^{\Uparrow} \circ R^{\odot}$ is a right closure operator.*
(3) *$R^{\odot} \circ R^{\Uparrow}$ is a right interior operator.*

Proof. (1) Since $R(x, y) \odot A(x) \odot (A(x) \Rightarrow B(x)) \leq R(x, y) \odot B(x)$, then $e^{\Uparrow}_{L^X}(A, B) \leq e^{\Uparrow}_{L^Y}(R^{\odot}(A), R^{\odot}(B))$. Since $(R(x, y) \Rightarrow A(y)) \odot (A(y) \Rightarrow B(y)) \leq$

$R(x,y) \Rightarrow B(y)$, then $e^{\Uparrow}_{L^Y}(A,B) \leq e^{\Uparrow}_{L^X}(R^{\Uparrow}(A), R^{\Uparrow}(B))$. By Theorem 3.5(1), we only show the following statement.

$$\begin{aligned} e^{\Uparrow}_{L^Y}(R^{\odot}(A),B) &= \bigwedge_{y\in Y}(R^{\odot}(A)(y) \Rightarrow B(y)) \\ &= \bigwedge_{y\in Y}\Big(\bigvee_{x\in X}(R(x,y)\odot A(x)) \Rightarrow B(y)\Big) \\ &= \bigwedge_{y\in Y}\bigwedge_{x\in X}\Big((R(x,y)\odot A(x)) \Rightarrow B(y)\Big) \\ &= \bigwedge_{x\in X}\bigwedge_{y\in Y}\Big(A(x) \Rightarrow (R(x,y) \Rightarrow B(y))\Big) \\ &= \bigwedge_{x\in X}\Big(A(x) \Rightarrow \bigwedge_{y\in Y}(R(x,y) \Rightarrow B(y))\Big) \\ &= \bigwedge_{x\in X}(A(x) \Rightarrow R^{\Uparrow}(B)(x)) \\ &= e^{\Uparrow}_{L^X}(A, R^{\Uparrow}(B)). \end{aligned}$$

(2) and (3) follow from Theorem 3.6.

Theorem 3.12 (1) $^{\odot}R$ *and* $R^{\uparrow}$ *are left isotone maps with a residuated connection* $(^{\odot}R, R^{\uparrow})$.
(2) $R^{\uparrow}\circ^{\odot} R$ *is a left closure operator.*
(3) $^{\odot}R\circ R^{\uparrow}$ *is a left interior operator.*

Proof. (1) Since $(A(x)\to B(x))\odot A(x)\odot R(x,y) \leq B(x)\odot R(x,y)$, then $e^{\uparrow}_{L^X}(A,B) \leq e^{\uparrow}_{L^Y}(^{\odot}R(A),^{\odot}R(B))$. Since $(A(y)\to B(y))\odot(R(x,y)\to A(y)) \leq R(x,y)\to B(y)$, then $e^{\uparrow}_{L^Y}(A,B) \leq e^{\uparrow}_{L^X}(R^{\uparrow}(A), R^{\uparrow}(B))$. By Theorem 3.3(1), we only show the following statement.

$$\begin{aligned} e^{\uparrow}_{L^Y}(^{\odot}R(A),B) &= \bigwedge_{y\in Y}(^{\odot}R(A)(y) \to B(y)) \\ &= \bigwedge_{y\in Y}\Big(\bigvee_{x\in X}(A(x)\odot R(x,y)) \to B(y)\Big) \\ &= \bigwedge_{y\in Y}\bigwedge_{x\in X}\Big(A(x)\odot (R(x,y)) \to B(y)\Big) \\ &= \bigwedge_{x\in X}\bigwedge_{y\in Y}\Big(A(x) \to (R(x,y) \to B(y))\Big) \\ &= \bigwedge_{x\in X}\Big(A(x) \to \bigwedge_{y\in Y}(R(x,y) \to B(y))\Big) \\ &= \bigwedge_{x\in X}(A(x) \to R^{\uparrow}(B)(x)) \\ &= e^{\uparrow}_{L^X}(A, R^{\uparrow}(B)) \end{aligned}$$

(2) and (3) follow from Corollary 3.7.

Example 3.13 Let $K = \{(x,y)\in R^2 \mid x>0\}$ be a set and we define an operation $\otimes : K\times K\to K$ as follows:

$$(x_1,y_1)\otimes(x_2,y_2) = (x_1x_2, x_1y_2+y_1).$$

Then $(K,\otimes)$ is a group with $e=(1,0)$, $(x,y)^{-1} = (\frac{1}{x}, -\frac{y}{x})$.

We have a positive cone $P = \{(a,b) \in R^2 \mid a = 1, b \geq 0\ \text{, or } a > 1\}$ because $P \cap P^{-1} = \{(1,0)\}$, $P \odot P \subset P$, $(a,b)^{-1} \odot P \odot (a,b) = P$ and $P \cup P^{-1} = K$. For $(x_1,y_1),(x_2,y_2) \in K$, we define

$$\begin{array}{ll}(x_1,y_1) \leq (x_2,y_2) & \Leftrightarrow (x_1,y_1)^{-1} \odot (x_2,y_2) \in P,\ (x_2,y_2) \odot (x_1,y_1)^{-1} \in P \\ & \Leftrightarrow x_1 < x_2 \ \text{ or } \ x_1 = x_2, y_1 \leq y_2.\end{array}$$

Then $(K, \leq \otimes)$ is a lattice-group. (ref. [1])

The structure $(L, \odot, \Rightarrow, \rightarrow, (\frac{1}{2},1),(1,0))$ is a generalized residuated lattice with strong negation where $\perp = (\frac{1}{2},1)$ is the least element and $\top = (1,0)$ is the greatest element from the following statements:

$$\begin{array}{ll}(x_1,y_1) \odot (x_2,y_2) & = (x_1,y_1) \otimes (x_2,y_2) \vee (\frac{1}{2},1) = (x_1x_2, x_1y_2 + y_1) \vee (\frac{1}{2},1), \\ (x_1,y_1) \Rightarrow (x_2,y_2) & = ((x_1,y_1)^{-1} \otimes (x_2,y_2)) \wedge (1,0) = (\frac{x_2}{x_1}, \frac{y_2 - y_1}{x_1}) \wedge (1,0), \\ (x_1,y_1) \rightarrow (x_2,y_2) & = ((x_2,y_2) \otimes (x_1,y_1)^{-1}) \wedge (1,0) = (\frac{x_2}{x_1}, -\frac{x_2y_1}{x_1} + y_2) \wedge (1,0).\end{array}$$

Furthermore, we have $(x,y) = (x,y)^{*\circ} = (x,y)^{\circ *}$ from:

$$(x,y)^* = (x,y) \Rightarrow (\frac{1}{2},1) = (\frac{1}{2x}, \frac{1-y}{x}),$$

$$(x,y)^{*\circ} = (\frac{1}{2x}, \frac{1-y}{x}) \rightarrow (\frac{1}{2},1) = (x,y).$$

Let $X = \{a,b,c\}$ and $Y = \{u,v\}$ be sets. Define $R \in L^{X \times Y}$ as

$$R = \begin{pmatrix} (1,0) & (\frac{5}{8},\frac{5}{2}) \\ (\frac{5}{7},\frac{30}{7}) & (\frac{5}{8},-\frac{5}{4}) \\ (\frac{1}{2},2) & (\frac{5}{6},\frac{10}{3}) \end{pmatrix}$$

For $A = ((\frac{2}{3},1),(\frac{1}{2},2),(\frac{2}{3},-1))^t$,

$$R^{\odot}(A) = ((\frac{2}{3},1),(\frac{5}{9},\frac{25}{6}))^t, \quad {}^{\odot}R(A) = ((\frac{2}{3},1),(\frac{5}{9},\frac{11}{9}))^t$$

$$R^{\rightarrow}(A) = ((\frac{3}{4},\frac{11}{4}),(\frac{15}{16},-\frac{25}{16}))^t, \quad R^{\Rightarrow}(A) = ((\frac{3}{4},\frac{9}{2}),(\frac{15}{16},\frac{9}{4}))^t$$

$$R^{\Uparrow}(R^{\odot}(A)) = ((\frac{2}{3},1),(\frac{8}{9},\frac{26}{3}),(\frac{2}{3},1))^t,$$

$$R^{\uparrow}({}^{\odot}R(A)) = ((\frac{2}{3},1),(\frac{8}{9},\frac{7}{3}),(\frac{2}{3},-1))^t$$

$$R^{\Leftarrow}(R^{\rightarrow}(A)) = ((\frac{2}{3},\frac{13}{3}),(\frac{2}{3},\frac{1}{3}),(\frac{2}{3},-1))^t,$$

$$R^{\leftarrow}(R^{\Rightarrow}(A)) = ((\frac{2}{3},\frac{85}{24}),(\frac{2}{3},-\frac{5}{24}),(\frac{2}{3},\frac{1}{6}))^t.$$

By Theorems 3.11 and 3.12, $R^{\Uparrow} \circ R^{\odot}$ is a right closure operator and $R^{\odot} \circ R^{\Uparrow}$ is a right interior operator. Moreover, $R^{\uparrow} \circ^{\odot} R$ is a left closure operator and ${}^{\odot}R \circ R^{\uparrow}$ is a left interior operator.

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