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A study on concave functions

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Abstract

A real-valued function f on a closed interval I is said to be concave if $tf(x) + (1-t)f(y) \leq f(tx + (1-x)y)$ for all $x, y \in I$ and for all $t \in [0, 1]$. In this paper, we present sufficient conditions for being a concave function.

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1 Introduction

A real-valued function f on a closed interval I is said to be *concave* if

$$f(x) + (1-t)f(y) \leq f(tx + (1-t)y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

For any continuous real-valued function f on a closed interval I, we obtain that f is *concave* if and only if

$$\frac{f(x) + f(y)}{2} \leq f\left(\frac{x+y}{2}\right)$$

for all $x, y \in I$.

In 2012, Sulaiman [1] gave some properties concerning operations on concave functions. In this paper, we present sufficient conditions for being a concave function.

2 Results

Theorem 2.1. *Let $f_1, f_2, ..., f_n$ be continuous real-valued functions on a closed interval I such that $\prod_{i=1}^{n} f_i \geq 0$ and*

$$\sqrt[n]{\frac{\prod_{i=1}^{n} f_i(x) + \prod_{i=1}^{n} f_i(y)}{2}} \leq \min_{i=1,2,...n} \left\{ f_i \left(\frac{x+y}{2} \right) \right\}$$

for all $x, y \in I$. Then $\prod_{i=1}^{n} f_i$ is concave.

Proof. Let $x, y \in I$. It follows that

$$f_j \left(\frac{x+y}{2} \right) = \min_{i=1,2,...n} \left\{ f_i \left(\frac{x+y}{2} \right) \right\}$$

for some $j \in \{1, 2, ...n\}$. Then

$$\begin{aligned} \frac{\prod_{i=1}^{n} f_i(x) + \prod_{i=1}^{n} f_i(y)}{2} &\leq \left[\min_{i=1,2,...n} \left\{ f_i \left(\frac{x+y}{2} \right) \right\} \right]^n \\ &= \left[f_j \left(\frac{x+y}{2} \right) \right]^n \\ &= \prod_{i=1}^{n} f_j \left(\frac{x+y}{2} \right) \\ &\leq \prod_{i=1}^{n} f_i \left(\frac{x+y}{2} \right). \end{aligned}$$

We note that $\prod_{i=1}^{n} f_i$ is a continuous real-valued function on I. It follows that $\prod_{i=1}^{n} f_i$ is concave. □

Theorem 2.2. *Let $f_1, f_2, ..., f_n$ and g be continuous real-valued functions on a closed interval I such that $g > 1$ and*

$$\sqrt[n]{\log_{g\left(\frac{x+y}{2}\right)} \frac{g^{f_1 f_2 \cdots f_n}(x) + g^{f_1 f_2 \cdots f_n}(y)}{2}} \leq \min_{i=1,2,...n} \left\{ f_i \left(\frac{x+y}{2} \right) \right\}$$

for all $x, y \in I$. *Then* $g^{f_1 f_2 \dots f_n}$ *is concave.*

Proof. Let $x, y \in I$. It follows that

$$f_j\left(\frac{x+y}{2}\right) = \min_{i=1,2,\dots n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}$$

for some $j \in \{1, 2, \dots n\}$. Then

$$\begin{aligned}\log_{g\left(\frac{x+y}{2}\right)} \frac{g^{f_1 f_2 \dots f_n}(x) + g^{f_1 f_2 \dots f_n}(y)}{2} &\leq \left[\min_{i=1,2,\dots n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}\right]^n \\ &= \left[f_j\left(\frac{x+y}{2}\right)\right]^n \\ &\leq f_1 f_2 \dots f_n\left(\frac{x+y}{2}\right).\end{aligned}$$

Then

$$g^{f_1 f_2 \dots f_n}\left(\frac{x+y}{2}\right) \leq \frac{g^{f_1 f_2 \dots f_n}(x) + g^{f_1 f_2 \dots f_n}(y)}{2}.$$

We note that $g^{f_1 f_2 \dots f_n}$ is a continuous real-valued function on I. It follows that $g^{f_1 f_2 \dots f_n}$ is concave. □

Theorem 2.3. *Let* $f_1, f_2, \dots, f_n$ *be continuous real-valued functions on a closed interval* I *such that*

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{f_i(x) + f_i(y)}{2}\right) \leq \min_{i=1,2,\dots n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}$$

for all $x, y \in I$. *Then* $\sum_{i=1}^{n} f_i$ *is concave.*

Proof. Let $x, y \in I$. It follows that

$$f_j\left(\frac{x+y}{2}\right) = \min_{i=1,2,\dots n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}$$

for some $j \in \{1, 2, \dots n\}$. Then

$$\begin{aligned}\frac{\sum_{i=1}^{n} f_i(x) + \sum_{i=1}^{n} f_i(y)}{2} &= \sum_{i=1}^{n} \left(\frac{f_i(x) + f_i(y)}{2} \right) \\ &\leq n \min_{i=1,2,\dots n} \left\{ f_i \left(\frac{x+y}{2} \right) \right\} \\ &= n f_j \left(\frac{x+y}{2} \right) \\ &= \sum_{i=1}^{n} f_j \left(\frac{x+y}{2} \right) \\ &\leq \sum_{i=1}^{n} f_i \left(\frac{x+y}{2} \right).\end{aligned}$$

We note that $\sum_{i=1}^{n} f_i$ is a continuous real-valued function on I. It follows that $\sum_{i=1}^{n} f_i$ is concave. □

Corollary 2.4. *Let $g_1, g_2, ..., g_n$ be continuous real-valued functions on a closed interval I such that*

$$\frac{1}{n} \sum_{i=1}^{n} (-1)^{i+1} \left(\frac{g_i(x) + g_i(y)}{2} \right) \leq \min_{i=1,2,\dots n} \left\{ (-1)^{i+1} g_i \left(\frac{x+y}{2} \right) \right\}$$

for all $x, y \in I$. Then $\sum_{i=1}^{n} (-1)^{i+1} g_i$ is concave.

References

[1] W. T. Sulaiman, Some operations on convex and concave functions, Eng. Math. Lett., 2012, **1**(1), 58–64.

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