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A study on convex functions

Banyat Sroysang

Department of Mathematics and Statistics,
Faculty of Science and Technology,
Thammasat University, Pathumthani 12121 Thailand
banyat@mathstat.sci.tu.ac.th

Abstract

In this paper, we present sufficient conditions for being a convex function.

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1 Introduction

For any real-valued function f on a closed interval I, we say that
(1) f is *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and all $t \in [0, 1]$, and
(2) f is *midpoint convex* if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

for all $x, y \in I$.

By Sierpinski Theorem, a real-valued Lebesgue measurable function that is midpoint convex will be convex.

In 2012, Sulaiman [1] gave some properties concerning operations on convex functions. In this paper, we present sufficient conditions for being a convex function.

2 Results

Theorem 2.1. *Let $f_1, f_2, ..., f_n$ be real-valued Lebesgue measurable functions on a closed interval I such that $\prod_{i=1}^{n} f_i \geq 0$ and*

$$\max_{i=1,2,...n}\left\{f_i\left(\frac{x+y}{2}\right)\right\} \leq \sqrt[n]{\frac{\prod_{i=1}^{n} f_i(x) + \prod_{i=1}^{n} f_i(y)}{2}}$$

for all $x, y \in I$. Then $\prod_{i=1}^{n} f_i$ is convex.

Proof. Let $x, y \in I$. It follows that

$$f_j\left(\frac{x+y}{2}\right) = \max_{i=1,2,...n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}$$

for some $j \in \{1, 2, ...n\}$. Then

$$\begin{aligned}
\prod_{i=1}^{n} f_i\left(\frac{x+y}{2}\right) &\leq \prod_{i=1}^{n} f_j\left(\frac{x+y}{2}\right) \\
&= \left[f_j\left(\frac{x+y}{2}\right)\right]^n \\
&= \left[\max_{i=1,2,...n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}\right]^n \\
&\leq \frac{\prod_{i=1}^{n} f_i(x) + \prod_{i=1}^{n} f_i(y)}{2}.
\end{aligned}$$

This implies that $\prod_{i=1}^{n} f_i$ is midpoint convex.

We note that $\prod_{i=1}^{n} f_i$ is a real-valued Lebesgue measurable function on I. By Sierpinski Theorem, we obtain that $\prod_{i=1}^{n} f_i$ is convex. □

Theorem 2.2. *Let $f_1, f_2, ..., f_n$ and g be real-valued Lebesgue measurable functions on a closed interval I such that $g > 1$ and*

$$\max_{i=1,2,\ldots n}\left\{f_i\left(\frac{x+y}{2}\right)\right\} \le \sqrt[n]{\log_{g\left(\frac{x+y}{2}\right)} \frac{g^{f_1f_2\cdots f_n}(x)+g^{f_1f_2\cdots f_n}(y)}{2}}$$

for all $x, y \in I$. Then $g^{f_1f_2\cdots f_n}$ is convex.

Proof. Let $x, y \in I$. It follows that

$$f_j\left(\frac{x+y}{2}\right) = \max_{i=1,2,\ldots n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}$$

for some $j \in \{1, 2, \ldots n\}$. Then

$$\begin{aligned} f_1f_2\ldots f_n\left(\frac{x+y}{2}\right) &\le \left[f_j\left(\frac{x+y}{2}\right)\right]^n \\ &= \left[\max_{i=1,2,\ldots n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}\right]^n \\ &\le \log_{g\left(\frac{x+y}{2}\right)} \frac{g^{f_1f_2\cdots f_n}(x)+g^{f_1f_2\cdots f_n}(y)}{2}. \end{aligned}$$

Then $g^{f_1f_2\cdots f_n}\left(\frac{x+y}{2}\right) \le \frac{g^{f_1f_2\cdots f_n}(x)+g^{f_1f_2\cdots f_n}(y)}{2}$. This implies that $g^{f_1f_2\cdots f_n}$ is midpoint convex.

We note that $g^{f_1f_2\cdots f_n}$ is a real-valued Lebesgue measurable function on I. By Sierpinski Theorem, we obtain that $g^{f_1f_2\cdots f_n}$ is convex. □

Theorem 2.3. *Let $f_1, f_2, \ldots, f_n$ be real-valued Lebesgue measurable functions on a closed interval I such that*

$$\max_{i=1,2,\ldots n}\left\{f_i\left(\frac{x+y}{2}\right)\right\} \le \frac{1}{n}\sum_{i=1}^{n}\left(\frac{f_i(x)+f_i(y)}{2}\right)$$

for all $x, y \in I$. Then $\sum_{i=1}^{n} f_i$ is convex.

Proof. Let $x, y \in I$. It follows that

$$f_j\left(\frac{x+y}{2}\right) = \max_{i=1,2,\ldots n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}$$

for some $j \in \{1, 2, ...n\}$. Then

$$\begin{aligned}
\sum_{i=1}^{n} f_i\left(\frac{x+y}{2}\right) &\le \sum_{i=1}^{n} f_j\left(\frac{x+y}{2}\right) \\
&= n f_j\left(\frac{x+y}{2}\right) \\
&= n \max_{i=1,2,\ldots n}\left\{f_i\left(\frac{x+y}{2}\right)\right\} \\
&\le \sum_{i=1}^{n}\left(\frac{f_i(x)+f_i(y)}{2}\right) \\
&= \frac{\sum_{i=1}^{n} f_i(x) + \sum_{i=1}^{n} f_i(y)}{2}.
\end{aligned}$$

This implies that $\sum_{i=1}^{n} f_i$ is midpoint convex.

We note that $\sum_{i=1}^{n} f_i$ is a real-valued Lebesgue measurable function on I. By Sierpinski Theorem, we obtain that $\sum_{i=1}^{n} f_i$ is convex. □

Corollary 2.4. *Let $g_1, g_2, ..., g_n$ be real-valued Lebesgue measurable functions on a closed interval I such that*

$$\max_{i=1,2,\ldots n}\left\{(-1)^{i+1} g_i\left(\frac{x+y}{2}\right)\right\} \le \frac{1}{n}\sum_{i=1}^{n}(-1)^{i+1}\left(\frac{g_i(x)+g_i(y)}{2}\right)$$

for all $x, y \in I$. Then $\sum_{i=1}^{n}(-1)^{i+1} g_i$ is convex.

References

[1] W. T. Sulaiman, Some operations on convex and concave functions, Eng. Math. Lett., 2012, **1**(1), 58–64.

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