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A study on logarithmically concave functions

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Abstract

For any positive real-valued function f on a closed interval, we say that (1) f is concave if $-f$ is convex, and (2) f is logarithmically concave if $\log f$ is concave. In this paper, we present sufficient conditions for being a logarithmically concave function.

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1 Introduction

A real-valued function f on a closed interval I is said to be *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

For any positive real-valued function f on a closed interval, we say that (1) f is *concave* if $-f$ is convex, and (2) f is *logarithmically concave* if $\log f$ is concave.

In 2012, Sulaiman [1] gave some properties concerning operations on concave functions.

For any continuous positive real-valued function f on a closed interval I, we obtain that f is logarithmically concave if and only if

$$\frac{\log f(x) + \log f(y)}{2} \leq \log f\left(\frac{x+y}{2}\right)$$

for all $x, y \in I$.

In this paper, we present sufficient conditions for being a logarithmically convex function.

2 Results

Theorem 2.1. *Let $f_1, f_2, ..., f_n$ be continuous positive real-valued functions on a closed interval I such that*

$$\sqrt[2n]{\prod_{i=1}^{n} f_i(x) f_i(y)} \leq \min_{i=1,2,...n} \left\{ f_i\left(\frac{x+y}{2}\right) \right\}$$

for all $x, y \in I$. Then $\prod_{i=1}^{n} f_i$ is logarithmically concave.

Proof. Let $x, y \in I$. It follows that

$$f_j\left(\frac{x+y}{2}\right) = \min_{i=1,2,...n} \left\{ f_i\left(\frac{x+y}{2}\right) \right\}$$

for some $j \in \{1, 2, ...n\}$. Then

$$\begin{aligned}
\frac{\log \prod_{i=1}^{n} f_i(x) + \log \prod_{i=1}^{n} f_i(y)}{2} &= \frac{1}{2} \log \prod_{i=1}^{n} f_i(x) f_i(y) \\
&= \log \sqrt{\prod_{i=1}^{n} f_i(x) f_i(y)} \\
&\leq \log \left[\min_{i=1,2,...n} \left\{ f_i\left(\frac{x+y}{2}\right) \right\} \right]^n \\
&= n \log \min_{i=1,2,...n} \left\{ f_i\left(\frac{x+y}{2}\right) \right\} \\
&= n \log f_j\left(\frac{x+y}{2}\right) \\
&= \sum_{i=1}^{n} \log f_j\left(\frac{x+y}{2}\right) \\
&\leq \sum_{i=1}^{n} \log f_i\left(\frac{x+y}{2}\right) \\
&= \log \prod_{i=1}^{n} f_i\left(\frac{x+y}{2}\right).
\end{aligned}$$

We note that $\prod_{i=1}^{n} f_i$ is a continuous positive real-valued function on I. Therefore $\prod_{i=1}^{n} f_i$ is logarithmically concave. □

Theorem 2.2. *Let $f_1, f_2, ..., f_n$ and g be continuous positive real-valued functions on a closed interval I such that $g > 1$ and*

$$\sqrt[n]{\frac{1}{2}\log_{g\left(\frac{x+y}{2}\right)} g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)} \leq \min_{i=1,2,\ldots n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}$$

for all $x, y \in I$. Then $g^{f_1 f_2 \cdots f_n}$ is logarithmically concave.

Proof. Let $x, y \in I$. It follows that

$$f_j\left(\frac{x+y}{2}\right) = \min_{i=1,2,\ldots n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}$$

for some $j \in \{1, 2, ...n\}$. Then

$$\begin{aligned}\log_{g\left(\frac{x+y}{2}\right)} \sqrt{g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)} &= \frac{1}{2}\log_{g\left(\frac{x+y}{2}\right)} g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y) \\ &\leq \left[\min_{i=1,2,\ldots n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}\right]^n \\ &= \left[f_j\left(\frac{x+y}{2}\right)\right]^n \\ &\leq f_1 f_2 \ldots f_n\left(\frac{x+y}{2}\right).\end{aligned}$$

Then

$$\sqrt{g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)} \leq g^{f_1 f_2 \cdots f_n}\left(\frac{x+y}{2}\right).$$

Then

$$\begin{aligned}\frac{\log g^{f_1 f_2 \cdots f_n}(x) + \log g^{f_1 f_2 \cdots f_n}(y)}{2} &= \frac{1}{2}\log g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y) \\ &= \log \sqrt{g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)} \\ &\leq \log g^{f_1 f_2 \cdots f_n}\left(\frac{x+y}{2}\right).\end{aligned}$$

We note that $g^{f_1 f_2 \cdots f_n}$ is a continuous positive real-valued function on I. Therefore $g^{f_1 f_2 \cdots f_n}$ is logarithmically concave. $\square$

A real-valued function f on a closed interval I is said to be *quasi−concave* if

$$f(tx + (1-t)y) \geq \min\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in [0,1]$.

Moreover, any positive logarithmically concave function is also quasi-concave.

Corollary 2.3. *Let $f_1, f_2, ..., f_n$ be continuous positive real-valued functions on a closed interval I such that*

$$\sqrt[2n]{\prod_{i=1}^{n} f_i(x) f_i(y)} \leq \min_{i=1,2,\ldots n} \left\{ f_i\left(\frac{x+y}{2}\right) \right\}$$

for all $x, y \in I$. Then $\prod_{i=1}^{n} f_i$ is quasi-concave.

Corollary 2.4. *Let $f_1, f_2, ..., f_n$ and g be continuous positive real-valued functions on a closed interval I such that $g > 1$ and*

$$\sqrt[n]{\frac{1}{2} \log_{g\left(\frac{x+y}{2}\right)} g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)} \leq \min_{i=1,2,\ldots n} \left\{ f_i\left(\frac{x+y}{2}\right) \right\}$$

for all $x, y \in I$. Then $g^{f_1 f_2 \cdots f_n}$ is quasi-concave.

References

[1] W. T. Sulaiman, Some operations on convex and concave functions, Eng. Math. Lett., 2012, **1**(1), 58–64.

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