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A study on logarithmically convex functions

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Abstract

A positive real-valued function f on a closed interval is said to be logarithmically convex if $\log f$ is convex. In this paper, we present sufficient conditions for being a logarithmically convex function.

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1 Introduction

Let I be a closed interval in $\mathbb{R}$. A real-valued function f on I is said to be *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

A positive real-valued function f on I is said to be *logarithmically convex* if $\log f$ is convex.

A real-valued function f on I is said to be *midpoint convex* if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for all $x, y \in I$.

By Sierpinski Theorem, for any positive real-valued Lebesgue measurable function f, if $\log f$ is midpoint convex then f is logarithmically convex.

In 2012, Sulaiman [1] gave some properties concerning operations between a convex function and a logarithmically convex function. In this paper, we present sufficient conditions for being a logarithmically convex function.

2 Results

Theorem 2.1. *Let $f_1, f_2, ..., f_n$ be positive real-valued Lebesgue measurable functions on a closed interval I such that*

$$\max_{i=1,2,...n}\left\{f_i\left(\frac{x+y}{2}\right)\right\} \leq \sqrt[2n]{\prod_{i=1}^{n} f_i(x)f_i(y)}$$

for all $x, y \in I$. Then $\prod_{i=1}^{n} f_i$ is logarithmically convex.

Proof. Let $x, y \in I$. It follows that

$$f_j\left(\frac{x+y}{2}\right) = \max_{i=1,2,...n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}$$

for some $j \in \{1, 2, ...n\}$. Then

$$\begin{aligned}
\log\prod_{i=1}^{n} f_i\left(\frac{x+y}{2}\right) &= \sum_{i=1}^{n}\log f_i\left(\frac{x+y}{2}\right) \\
&\leq \sum_{i=1}^{n}\log f_j\left(\frac{x+y}{2}\right) \\
&= n\log f_j\left(\frac{x+y}{2}\right) \\
&= n\log\max_{i=1,2,...n}\left\{f_i\left(\frac{x+y}{2}\right)\right\} \\
&= \log\left[\max_{i=1,2,...n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}\right]^n \\
&\leq \log\sqrt{\prod_{i=1}^{n} f_i(x)f_i(y)} \\
&= \frac{1}{2}\log\prod_{i=1}^{n} f_i(x)f_i(y) \\
&= \frac{\log\prod_{i=1}^{n} f_i(x) + \log\prod_{i=1}^{n} f_i(y)}{2}.
\end{aligned}$$

Then $\log\prod_{i=1}^{n} f_i$ is midpoint convex. We note that $\prod_{i=1}^{n} f_i$ is a positive real-valued Lebesgue measurable function on I. By Sierpinski Theorem, we obtain that $\prod_{i=1}^{n} f_i$ is logarithmically convex. $\square$

Corollary 2.2. *Let $f_1, f_2, ..., f_n$ be continuous positive real-valued functions on a closed interval I such that*

$$\max_{i=1,2,...n}\left\{f_i\left(\frac{x+y}{2}\right)\right\} \le \sqrt[2n]{\prod_{i=1}^{n} f_i(x)f_i(y)}$$

for all $x, y \in I$. Then $\prod_{i=1}^{n} f_i$ is logarithmically convex.

Theorem 2.3. *Let $f_1, f_2, ..., f_n$ and g be positive real-valued Lebesgue measurable functions on a closed interval I such that $g > 1$ and*

$$\max_{i=1,2,...n}\left\{f_i\left(\frac{x+y}{2}\right)\right\} \le \sqrt[n]{\frac{1}{2}\log_{g\left(\frac{x+y}{2}\right)} g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)}$$

for all $x, y \in I$. Then $g^{f_1 f_2 \cdots f_n}$ is logarithmically convex.

Proof. Let $x, y \in I$. It follows that

$$f_j\left(\frac{x+y}{2}\right) = \max_{i=1,2,...n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}$$

for some $j \in \{1, 2, ...n\}$. Then

$$\begin{aligned} f_1 f_2 \cdots f_n\left(\frac{x+y}{2}\right) &\le \left[f_j\left(\frac{x+y}{2}\right)\right]^n \\ &= \left[\max_{i=1,2,...n}\left\{f_i\left(\frac{x+y}{2}\right)\right\}\right]^n \\ &\le \frac{1}{2}\log_{g\left(\frac{x+y}{2}\right)} g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y) \\ &= \log_{g\left(\frac{x+y}{2}\right)} \sqrt{g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)}. \end{aligned}$$

Then

$$g^{f_1 f_2 \cdots f_n}\left(\frac{x+y}{2}\right) \le \sqrt{g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)}.$$

Then

$$\log g^{f_1 f_2 \cdots f_n}\left(\frac{x+y}{2}\right) \leq \log \sqrt{g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)}$$
$$= \frac{1}{2} \log g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)$$
$$= \frac{\log g^{f_1 f_2 \cdots f_n}(x) + \log g^{f_1 f_2 \cdots f_n}(y)}{2}.$$

Thus $\log g^{f_1 f_2 \cdots f_n}$ is midpoint convex. We note that $g^{f_1 f_2 \cdots f_n}$ is a positive real-valued Lebesgue measurable function on I. By Sierpinski Theorem, we obtain that $g^{f_1 f_2 \cdots f_n}$ is logarithmically convex. □

Corollary 2.4. *Let $f_1, f_2, ..., f_n$ and g be continuous positive real-valued functions on a closed interval I such that $g > 1$ and*

$$\max_{i=1,2,\ldots n} \left\{ f_i \left(\frac{x+y}{2}\right) \right\} \leq \sqrt[n]{\frac{1}{2} \log_{g\left(\frac{x+y}{2}\right)} g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)}$$

for all $x, y \in I$. Then $g^{f_1 f_2 \cdots f_n}$ is logarithmically convex.

References

[1] W. T. Sulaiman, Some operations on convex and concave functions, Eng. Math. Lett., 2012, **1**(1), 58–64.

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