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DEFORMATION PROPERTIES OF G-ANR SPACES

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Abstract

In this paper we shall establish some deformation properties of G-ANR spaces generated by the notion of G-ANR divisors. This concept in the theory of retracts was established by D. M. Hyman in [8].

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1 Introduction

The property of a space B such that if it is closed embedded into a ANR space X then the space X/B obtained of collapsing B to a point, is a ANE, is established in [8], and it is is said that B is a ANR divisor. D. M. Hyman gave several characterizations of these spaces in his paper and in [9].

We shall give the equivariant analogous of these results, when the acting group is a compact.

2 Preliminary Notes

Here and in what follows G will always denote a Hausdorff compact group. By a G-space we mean a topological space where G acts continuously. The basic ideas of G-spaces can be found in [5], [6], [10]. A subset B of a G-space is said invariant or G-subset if $GB = B$. The subset $G(x) = \{gx \in X | g \in G\}$ is called the orbit of x. These subsets make a partition of X and we obtain a new space called the orbit space, such is denoted like X/G. By a map $f : X \to Y$ of a space X into a space Y, we mean a continuous function from X to Y. If X and Y are G-spaces, then a map $f : X \to Y$ is a equivariant map or G-map satisfying $f(gx) = gf(x)$; that is, f commute with the action. If $f(gx) = f(x)$, then f is said invariant map.

Let X be a metrizable G-space. A metric d over X is said invariant if each transition is a d-isometry and d is compatible with the topology of X.

The equivariant definitions of $A(N)E$ and $A(N)R$ are similar those classic definitions, and the reader can see, for instance, [1], [2], [3]. We consider the class of spaces G-M of all metrizable G-spaces. Since G is compact, by [10] each space belongs to G-M has an invariant metric.

A couple (X, A) is a G-pair if X is a G-space and A is an invariant closed subset of X.

A G-space Y is called a G-ANE (for the class G-M) (notation: $Y \in G$-ANE), if for any G-pair (X, A) with $X \in G$-M and any G-map $f : A \to Y$, there exist an invariant neighborhood U of A in X and a G-map $\psi : U \to Y$ such that $\psi|_A = f$. The map ψ is called a G-extension of f over U. If in addition we can always take $U = X$, then we say that Y is a G-AE (notation: $Y \in G$-AE).

Let A be an invariant closed subset of X. Then A is called equivariant neighborhood retract of X if there exists a G-map $r : U \to A$ with U an invariant neighborhood of A in X, such that $r|_A = id_A$ where id_A is the identity map on A. The G-map r is called a G-retraction of U onto A. If $U = X$ then A is called G-retract or equivariant retract of X.

Let Y be a G-space. Then Y is called a G-ANR (notation: $Y \in G$-ANR) provided $Y \in G$-M, and for any G-space X from G-M, where Y is embedded as invariant closed subset, Y is a equivariant neighborhood retract of X. If in addition Y is G-retract of X, then we say that Y is a G-AR (notation: $Y \in G$-AR).

Let X, Y be G-spaces and $\{h_t : X \to Y | t \in I\}$ be a G-maps family with indexing set the unit interval $I = [0, 1]$. The family $\{h_t | t \in I\}$ is called a G-homotopy from h_0 to h_1, if the function $H : X \times I \to Y$ defined by $H(x, t) = h_t(x)$ for every $x \in X$ and $t \in I$, is a G-map. Here I has the trivial action and $X \times I$ the diagonal action. The G-map H is called G-homotopy too. Frequently, we use the notation h_t, $t \in I$, to represent the G-homotopy $\{h_t | t \in I\}$ from h_0 to h_1.

Let f_0, $f_1 : X \to Y$ be two G-maps. They are said G-homotopic if there exists a G-homotopy f_t, $t \in I$ from f_0 to f_1. In addition, the relation being

G-homotopic is an equivalence relation and we have a category of G-spaces and G-homotopy classes of mappings. We write $f_0 \overset{G}{\sim} f_1$ if f_0 and f_1 are G-homotopic.

Let A, B be any two invariant subsets of G-space X. B is said to be G-deformable into A over X if the identity G-map $id : B \to B$ is G-homotopic in X to a G-map of B into A. That is, we require a G-homotopy f_t, $t \in I$, called a G-deformation, such that $f_0(b) = b$ for each $b \in B$, and $f_1(B) \subset A$. If we have $B = X$, we omit "over X " and say simply that X is G-deformable into A.

An G-map $f : X \to Y$ G-homotopic to a constant G-map is called nullhomotopic G-map. Call a G-space G-contractible if $id : Y \to Y$ is nullhomotopic G-map.

An invariant subset A of a G-space X is a strong neighborhood G-deformation retract of X if there exist an invariant neighborhood U of A in X and a G-homotopy f_t, $t \in I$ from f_0 into f_1, such that f_0 is the inclusion $U \hookrightarrow X$, f_1 is a G-retraction of U onto A, and $f_t(a) = a$ for all $a \in A$ and $t \in I$.

Let $f : X \to Y$ a G-map. We say that a G-map $h : Y \to X$ is a right G-homotopy inverse of f if the composition fh is G-homotopic to the identity in Y. Analogously, we define a left G-homotopy inverse of f. The G-map $f : X \to Y$ is called a G-homotopy equivalence if there exist a G-map $h : Y \to X$ such that both $fh \overset{G}{\sim} id_Y$ and $hf \overset{G}{\sim} id_X$. We shall say that a G-space X is G-homotopically dominated by a G-space Y, if there exist a G-map $f : X \to Y$ such that f have a left G-homotopy inverse.

We shall say that two G-spaces X and Y have the same G-homotopy type if we can find two G-maps $f : X \to Y$ and $h : Y \to X$ such that the compositions fh and hf are G-homotopic to the appropiate identity.

Let (Y, B) be a G-pair. We shall say that B is a strong neighborhood G-deformation retract of Y if there exists an invariant neighborhood W of B and a G-homotopy $h_t : W \to Y$, $t \in I$, where I have the trivial action of G, such that h_0 is the inclusion of B in Y, and h_1 is a G-retraction of W over B and $h(b, t) = b$ for all $b \in B$ and for all $t \in I$.

We notice that in general a metrizable G-ANE space Y need not be a G-ANR, because it may not belong to the class G-M. But if $Y \in G$-M and $Y \in G$-ANE, then it is easy to see that, $Y \in G$-ANR. We constantly refer to the following result, whose proof can be found in ([3], Theorem 14).

Theorem 2.1 *Let X a metric G-space. Then, X is a G-ANE if and only if it is G-ANR.*

It is well known the following result which is frequently used in this document.

Theorem 2.2 *Let W be an invariant open subset of a G-space Y. If $Y \in G$-ANE, then W is a G-ANE.*

Other result used in this work corresponds to the equivariant theorem of K. Borsuk about of the homotopy extension property and ANR's spaces (see [[1], Theorem 5]). Firs we define the G-homotopy extension property.

Definition 2.3 *A G-pair (X, A) is said that has the G-homotopy extension property (abbreviated G-HEP) respect to a G-space Y is given G-maps $f : X \to Y$ and $H : A \times I \to Y$ such that $H(a, 0) = f(a)$ for all $a \in A$, then there exists a G-map $H^* : X \times I \to Y$ satisfying $H^*(a, t) = f(a)$ for all $a \in A$, $t \in I$ and $H^*(x, 0) = f(x)$ for all $x \in X$.*

Theorem 2.4 *Let (X, A) be a metric G-pair. Then (X, A) has the G-HEP respect to every G-ANR.*

Finally, we use the following application of the equivariant generalization of the Borsuk-Whitehead-Hanner theorem (see [4], Corollary 3.12). In the theory of retracts the readers can see [7] and [8].

Theorem 2.5 *Let (X, A) be a G-pair whit $X \in G$-$M \cap G$-ANE and $A \in G$-ANE. Then $X/A \in G$-ANE.*

3 G-ANR divisors

We will need the following results to introduce the definition of G-ANR divisor.

Theorem 3.1 *Let Y be a metric G-space, X be a G-space and $f : B \to X$ a G-map, where B is an invariant closed subset of Y. If X, $Y \in G$-ANE then, $Y \cup_f X \in G$-ANE if and only if $p(X)$ is a strong neighborhood G-deformation retract of $Y \cup_f X$.*

Proof. See ([4], Lemma 3.9). □

By G-$ANR(B)$ we shall denote the class of G-ANR containing to B as an invariant closed subspace.

Theorem 3.2 *Let B be a G-space, let $X \in G$-ANR and let $f : B \to X$ be a G-map. If there exists a G-space $Y_0 \in G$-$ANR(B)$ such that $Y_0 \cup_f X \in G$-ANE, then for every $Y \in G$-$ANR(B)$ we have $Y \cup_f X$ is a G-ANE.*

Proof. Let $p : Y \sqcup X \to Y \cup_f X$ be the canonical projection. To prove that $Y \cup_f X$ is a G-ANE, it suffices, by theorem 3.1, to show that $p(X)$ is a strong neighborhood G-deformation retract of $Y \cup_f X$.

Since $Y \in G$-ANR, by theorem 2.1, $Y \in G$-ANE. Then, the inclusion $i : B \to Y$ have a G-extension $\phi : U \to Y$ over an invariant neighborhood U of B in Y_0. Since $Y \in G$-ANR, by theorem 2.1, $Y \in G$-ANE. Let $q : Y_0 \sqcup X \to Y_0 \cup_f X$ be the natural projection; then, we have that $q(U \sqcup X)$ is open in $Y_0 \cup_f X$ and, like U, is a G-ANE. By the theorem 3.1, there exists a strong neighborhood G-deformation retraction $h : W \times I \to q(U \sqcup X)$, where W is an invariant neighborhood of $q(X)$ in $q(U \sqcup X)$ and, accordingly, is open in $Y_0 \cup_f X$. Since $q^{-1}(W) \cap Y_0$ is invariant open set in Y_0 and, by theorem 2.1, $X_0 \in G$-ANE; by the theorem 2.2, we have that $q^{-1}(W) \cap Y_0 \in G$-ANE. Thus, the inclusion $j : B \to q^{-1}(W) \cap Y_0$ have a G-extension $\psi : V \to q^{-1}(W) \cap Y_0$, where V is an invariant neighborhood of B in Y. Also, $V \in G$-ANE, by the theorem 2.2. It follows that there exists an invariant neighborhood D of A in U and a G-deformation $s : D \times I \to V$ satisfying $s(b, t) = b$ for all $b \in B$, $t \in I$ and $s_1 = \phi\psi|_V$.

Let $\Psi : U \sqcup X \to Y \sqcup X$ be a G-map, defined by

$$\Psi(x) = \begin{cases} \phi(x), & \textit{if } \ x \in Y, \\ x, & \textit{if } \ x \in X, \end{cases}$$

Define a G-map $k : p(D \sqcup X) \times I \to Y \cup_f X$ by

$$k(z,t) = \begin{cases} ps_{2t}(p|Y)^{-1}(z), & \textit{if } z \in p(D) \textit{ and } \ 0 \le t \le 1/2, \\ p\Psi q^{-1} h_{2t-1} q\psi(p|Y)^{-1}(z), & \textit{if } z \in p(D) \textit{ and } \ 1/2 \le t \le 1, \\ z, & \textit{if } \ z \in p(X) \textit{ and } \ 0 \le t \le 1. \end{cases}$$

Then k is a strong neighborhood G-deformation retraction and concludes the proof. □

Corollary 3.3 *Let B be a metric G-space. If there exists a G-space $Y_0 \in G$-$ANR(B)$ such that $Y_0/B \in G$-ANE, then for every $Y \in G$-$ANR(B)$ we have Y/B is a G-ANE.*

When B is compact, then Y/B is metrizable [7]; applying this fact to corollary 3.3, we have

Theorem 3.4 *Let B be a compact metrizable G-space. If there exists a G-space $Y_0 \in G$-$ANR(B)$ such that $Y_0/B \in G$-ANR, then for every G-space $Y \in G$-$ANR(B)$, we have that Y/B is an G-ANR.*

Definition 3.5 *A G-space B is called a G-ANR divisor if it is metrizable and Y/B is a G-ANE for every G-space $Y \in G$-$ANR(B)$.*

Remark 3.6 *By the theorem 2.5, we note that a compact metrizable G-ANR G-space it will also be a G-ANR divisor.*

4 G-deformation neighborhood basis

Definition 4.1 *A G-space X is a strongly locally G-contractible at a point $x \in X$ if there is an invariant neighborhood V of x in X and a G-contraction k_t, of V into X such that $k_t(x) = x$, for all $t \in I$.*

By theorem 3.1 and previous definition, we obtain:

Lemma 4.2 *Let B be an invariant closed subset of a G-space $Y \in G$-ANR. Then $Y/B \in G$-ANE if and only if Y/B is strongly locally G-contractible at a point $p(B)$, where $p : Y \to Y/B$ is the canonical projection.*

Now, we shall introduce the notion of a G-deformation neighborhood basis with the purpose to establish condition for Y/B to be a strongly locally G-contractible at a point $p(B)$.

Definition 4.3 *Let (Y, B) be a G-pair. A sequence $\{U_n, h_n\}_{n\geq 1}$, is called a G-deformation neighborhood basis for B in Y if it satisfies*

(B1) *Each U_n is an invariant neighborhood of B in Y.*

(B2) *$\overline{U}_{n+1} \subset U_n$, for all $n \in N$.*

(B3) *For each invariant neighborhood V of B in Y, there exists $n \in N$ such that $U_n \subset V$.*

(B4) *Let $U_0 = Y$. Then, for all $n \geq 1$, $h_n : \overline{U}_n \times I \to \overline{U}_{n-1}$ is a G-deformation such that*

$$h_n(\overline{U_n} \times \{1\}) \subset \overline{U}_{n+1}$$

(B5) *If $m > n$, then $h_n(\overline{U}_m \times I) \subset \overline{U}_{m-1}$.*

Lemma 4.4 *Let (Y, B) be a G-pair. If B has a G-deformation neighborhood basis in Y, then Y/B is strongly locally G-contractible at a point $p(B)$, where $p : Y \to Y/B$ is the canonical projection.*

Proof. Let $\{U_n, h_n\}_{n\geq 1}$ a G-deformation neighborhood basis for B. For each n and for all $s \in I$, let $h_n^s : \overline{U}_n \to Y$ a G-map given by $h_n^s(x) = h_n(x, s)$. Now we define the G-map $h : \overline{U}_1 \times [0, \infty) \to Y$ by

$$h(x,t) = \begin{cases} h_1(x,t), & \textit{if} \;\; k = 0, \\ h_{k+1}^{t-k} \circ h_k^1 \circ h_{k-1}^1 \circ ... \circ h_2^1 \circ h_1^1(x), & \textit{if} \;\; k \geq 1. \end{cases}$$

where k is a non-negative integer such that $t \in [k, k+1]$.

Since the range of h_n^1 is contained in the domain of h_{n+1}^s for all n and s, composition is well defined.

We verify that is well-defined. Let $(x,t) \in \overline{U}_1 \times [0, \infty)$, where $t = k$, and k is a non-negative integer. Then $t \in [k-1, k] \cap [k, k+1]$. Thus,

$$h(x,t) = h_k^1 \circ h_{k-1}^1 \circ ... \circ h_2^1 \circ h_1^1(x)$$

and other hand

$$h(x,t) = h_{k+1}^0 \circ h_k^1 \circ h_{k-1}^1 \circ ... \circ h_2^1 \circ h_1^1(x) = h_{k+1}^0 \circ h(x,t).$$

But, h_n is a G-deformation for each n. Then,

$$h(x,t) = h_{k+1}^0 \circ h(x,t) = h_{k+1}(h(x,t), 0) = h(x,t).$$

So h is well-defined.

Also, h is continuous, since is continuous in each closed subset $\overline{U}_1 \times [k, k+1]$. In addition, is easily to see that is equivariant since is composition of G-maps.

Moreover, h has the following properties:

(1) For all $n > 1$, $h(\overline{U}_n \times [0, \infty)) \subset \overline{U}_{[n/2]}$ (here $[n/2]$ denotes the greatest integer less than or equal to $n/2$).

(2) For all $t \in [1, \infty)$, we have $h(\overline{U}_1 \times \{t\}) \subset \overline{U}_{[t]}$.

(3) $h(B \times [0, \infty)) \subset B$.

We verify the previous properties are satisfy.

(1) Let $(x,t) \in \overline{U_n} \times [0, \infty)$ and k a non negative integer such that $t \in [k, k+1]$.

   (i) If $k \geq [n/2]$, by (B2) we must consider that $x \in \overline{U}_1$. Then by (B4) and the definition of h_1 we have

$$h(x,t) \in \overline{U}_k \subset \overline{U}_{[n/2]}.$$

   (ii) If $k < [n/2]$ is trivial to check that $n - k - 1 \geq [n/2]$. Since $x \in \overline{U}_n$ and $n - k > k + 1$, we apply (B5) in the definition of h, and we obtain $h(x,t) \in U_{[n/2]}$, so the proof of (1) is finished.

(2) In this case, let $k \geq 1$, where $k \leq t \leq k+1$. Then $[t] = k$ and by (B4), h_{k+1} is a deformation into $\overline{U_k}$. We conclude applying the definition of h.

(3) By (B1) and (B2) follows that $B = \cap_{n=1}^{\infty} \overline{U}_n$. So, applying part (1) of this lemma, we obtain the desired.

We consider a homeomorphism f of $[0,1)$ onto $[0,\infty)$. Define a function

$$J : p(U_1) \times I \to Y/B,$$

by

$$J(x,t) = \begin{cases} p(h(p^{-1}(x), f(t))), & \textit{if } x \in p(U_1) \textit{ and } t < 1, \\ p(B), & \textit{if } x \in p(U_1) \textit{ and } t = 1. \end{cases}$$

It is clear that J is equivariant. Now, to see that is well-defined, it is necessary to check only in $p(B)$. Let b_1, $b_2 \in B$ and $t \in [0,1)$. Then, accordance with (3), $h(p^{-1}(p(b_1)), f(t)) \in B$. Thus, $p(h(p^{-1}(p(b_1)), f(t))) = J(p(b_1), t) = J(p(b_2), t)$ and J is well-defined.

Moreover, $J(p(B) \times I) = p(B)$.

In order that J to be continuous it is sufficient to show that J is continuous at the points of $p(U_1) \times \{1\}$ and $p(B) \times I$.

It is easy to see that each U_n is a satured set by p. Then, for each $n \in N$, $p(U_n)$ is a neighborhood of $p(B)$. Moreover, by (B3), $\{p(U_n) | n \geq 1\}$ is a neighborhood basis of $p(B)$ in Y/B. From here will prove the continuity of J.

Let $(w_0, 1) \in p(U_1) \times \{1\}$. Let V be a neighborhood of $J(w,1) = p(B)$. Then there exists a non-negative integer m such that $p(B) \in p(U_m) \subset V$. We consider $f^{-1}(m,\infty) \cup \{1\}$, which has the form $(r, 1]$, for some $r \in [0,1)$. Let $W = p(U_1) \times (r,1]$. Thus, W is a neighborhood of $(w_0, 1)$. We affirm that $J(W) \subset V$. In fact, let $(w,t) \in W$. The case $t = 1$ is trivial. If $t \in [0,1)$ then $J(w,t) = p(h(p^{-1}(w), f(t)))$; since $t \in f^{-1}(m,\infty)$ it follows that $f(t) \in (m,\infty)$ and by (2), $h(p^{-1}(w), f(t)) \in U_m$. So $J(w,t) = p(h(p^{-1}(w), f(t))) \in p(U_m) \subset V$.

At the same way, let $(p(B), t) \in p(B) \times I$ and V a neighborhood of $p(B)$ in Y/B. Then, there exists a neighborhood $p(U_m)$ of $p(B)$ in Y/B contained in V. We choose a non-negative integer k such that $[k/2] > m$. If we consider the neighborhood $W = p(U_k) \times I$ of $(p(B), t)$ and apply (1), then we have $J(W) \subset p(\overline{U}_{[k/2]}) \subset p(U_m) \subset V$. We conclude that J is continuous.

Finally, only is necessary to check that J is a contraction from a neighborhood of $p(B)$ into Y/B. If follows of the definition of J that $J|_{p(U_1)\times\{1\}} = p(B)$ and for each $z \in p(U_1)$, $J(z,0) = p(h(p^{-1}(z), 0)) = pp^{-1}(z) = z$.

This complete the proof. □

From the lemmas 4.2 and 4.4, we obtain:

Theorem 4.5 *Let B be an invariant closed subset of a G-space $Y \in G$-ANR. If B has a G-deformation neighborhood basis in Y then $Y/B \in G$-ANE.*

From corollary 3.3 and theorem 4.5, it follows

Corollary 4.6 *Let B be an invariant closed subset of a G-space $Y \in G$-ANR. If B has a G-deformation neighborhood basis in Y then B is a G-ANR divisor.*

5 Absolute neighborhood G-contractibility

Definition 5.1 *Let (Y, B) be a G-pair. B is said to be neighborhood G-contractible in Y if B is G-contractible in every invariant neighborhood U of B in Y.*

Observation 5.2 *A neighborhood G-contractible G-space is G-contractible.*

Definition 5.3 *A metric G-space B is said to be absolutely neighborhood G-contractible if B is neighborhood G-contractible in every $Y \in G$-$ANR(B)$.*

The next theorem characterizes the property of a metric G-space to be absolutely neighborhood G-contractible, through some weaker conditions.

Theorem 5.4 *Let B be a metric G-space. Then are equivalents,*

(a) There exists a $Y \in G$-$ANR(B)$ such that B is neighborhood G-contractible in Y.

(b) For each $Y \in G$-$ANR(B)$, we have that B is G-contractible in Y.

(c) B is absolutely neighborhood G-contractible.

Proof. Then we show that (a) implies (b). Let $Z \in G$-$ANR(B)$. Then the identity G-map $i : B \to B$ extends to a G-map $\varphi : U \to Z$, where U is an invariant neighborhood of B in Y. By (a) B is G-contractible in U under an invariant homotopy h_t. Hence the homotopy φh_t equivariantly contract B into Z.

Now, we show that (b) implies (c). Let U be an invariant neighborhood of B in Y. Then, by theorem 2.1 and theorem 2.2, U is a G-$ANR(B)$ and by (b) B is G-contractible into U. Therefore, B is absolutely neighborhood G-contractible.

Finally, the prove that (c) implies (a) is trivial. □

In this section we will show that every absolutely neighborhood G-contractible compactum is a G-ANR divisor. Before this fact it is necessary some previous theorems, the first of which is a characterization of absolute neighborhood G-contractibility. Then mention a corollary of an important result such that will represent a useful tool, the *Equivariant Extension Homotopy Theorem* 2.4.

Since any constant G-map on A can be extended a X, for the theorem 2.4 it follows that

Corollary 5.5 *Let (X, A) be a metric G-pair and f a nullhomotopic G-map from A into G-space $Y \in G$-ANR. Then f has a G-extension $F : X \to Y$.*

Theorem 5.6 *Let B be a metric G-space. Then, B is absolutely neighborhood G-contractible if and only if for every G-space $Y \in G$-$ANR(B)$, there exists an invariant neighborhood V of B in Y such that for every metric G-pair (X, A), each G-map $f : A \to \overline{V}$ has an equivariant extension $F : X \to Y$.*

Proof. First, suppose that B is absolutely neighborhood G-contractible and let $Y \in G$-$ANR(B)$. Let k_t an equivariant contraction of B over Y to a point b_0. Now we define a G-map $h : (Y \times \{0\}) \cup (B \times I) \cup (Y \times \{1\}) \to Y$ by

$$h(y,t) = \begin{cases} y, & \textit{if } y \in Y \textit{ and } t = 0, \\ k_t(y), & \textit{if } y \in B \textit{ and } t \in I, \\ b_0, & \textit{if } y \in Y \textit{ and } t = 1. \end{cases}$$

Since Y is a G-$ANR(B)$, h has an equivariant extension $H : W \to Y$, where W is an invariant open neighborhood of $(Y \times \{0\}) \cup (B \times I) \cup (Y \times \{1\})$ in $Y \times I$. Let V an invariant neighborhood of B in Y such that $\overline{V} \times I \subset W$. Hence $H|_{\overline{V} \times I}$ equivariantly contract $\overline{V}$ over Y to a point b_0. Let $f : A \to \overline{V}$ be any G-map and $J = H \circ (f \times id)$, where id is the identity G-map on I. Clearly, J is a G-homotopy of $A \times I$ in Y and it follows that f is G-nullhomotopic over Y, and by corollary 5.5 equivariantly extends on Y.

Conversely, let $Y \in G$-$ANR(B)$ and V an invariant neighborhood of B such that satisfies the property stated in the hypothesis. Then the G-map $f : (\overline{V} \times \{0\}) \cup (\overline{V} \times \{1\}) \to \overline{V}$ defined by $f(v, 0) = v$, $f(v, 1) = b_0$ has an equivariant extension $F : \overline{V} \times I \to Y$. Hence $\overline{V}$ is G-contractible over Y; in particular, B is contractible over Y and applying theorem 5.4 (b), we complete the proof. □

Lemma 5.7 *Let B be a compact absolutely neighborhood G-contractible metric G-space and $Y \in G$-$ANR(B)$. Then B has a G-deformation neighborhood basis.*

Proof. By the previous theorem there exists an invariant neighborhood U_1 of B in Y such that any G-map from an invariant closed subset of a metric G-space into $\overline{U_1}$ has an equivariant extension over Y. We may choose U_1 such that $d(x,B) < 1$ for all $x \in U_1$, where d is some metric on Y. Accordance with theorem 2.1 and theorem 2.2, we apply successively the theorem 5.6 obtaining a sequence of invariant neighborhoods $\{U_n\}_{n>1}$ of B such that

(1) $\overline{U}_n \subset U_{n-1}$;

(2) every G-map from an invariant closed subset of a metric G-space into $\overline{U}_n$ has an equivariant extension over U_{n-1};

(3) for all $x \in U_n$ we have that $d(x,B) < 1/n$.

By (1) and (3) it follows that the sequence $\{U_n\}_{n\geq 1}$ satisfies (B1)-(B3). Now, we verify (B4) and (B5). First, we choose a point $b_0 \in B$. For each positive integer n, define a G-map

$$f_n : (\overline{U}_n \backslash \overline{U}_{n+1}) \cup \overline{U}_{n+2} \to \overline{U}_{n+2}$$

by

$$f_n(x) = \begin{cases} b_0, & \textit{if } x \in (\overline{U}_n \backslash \overline{U}_{n+1}), \\ x & \textit{if } x \in \overline{U}_{n+2}. \end{cases}$$

From (2), f_n extends to an G-map $F_n : \overline{U}_n \to U_{n+1}$. Define a G-map by

$$j_n : (\overline{U}_{n+1} \times \{0\}) \cup (\overline{U}_{n+2} \times I) \cup (\overline{U}_{n+1} \times \{1\}) \to \overline{U}_{n+1}$$

by

$$j_n(x,t) = \begin{cases} x, & \textit{if } x \in \overline{U}_{n+1} \textit{ and } \ t = 0, \\ x & \textit{if } x \in \overline{U}_{n+2} \textit{ and } \ 0 \leq t \leq 1, \\ F_n(x) & \textit{if } x \in \overline{U}_{n+1} \textit{ and } \ t = 1. \end{cases}$$

At he same way, by (2), j_n extends to a G-map $J_n : \overline{U}_{n+1} \times I \to U_n$.

To finish, we define a G-map

$$k_n : (\overline{U}_n \times \{0\}) \cup (\overline{U}_{n+1} \times I) \cup (\overline{U}_n \times \{1\}) \to \overline{U}_n$$

as follows

$$k_n(x,t) = \begin{cases} x, & \textit{if } x \in \overline{U}_n \textit{ and } \ t = 0, \\ J_n(x,t) & \textit{if } x \in \overline{U}_{n+1} \textit{ and } \ 0 \leq t \leq 1, \\ F_n(x) & \textit{if } x \in \overline{U}_n \textit{ and } \ t = 1 \end{cases}$$

Again by (2), k_n has an equivariant extension $h_n : \overline{U}_n \times I \to U_{n-1}$ if $n > 1$; while k_1 extends to an equivariant map $h_1 : \overline{U}_1 \times I \to Y$. It is easily to see that the sequence $\{h_n | n \geq 1\}$ satisfies (B4)-(B5). So $\{(U_n, h_n)\}_{n\geq 1}$ is a G-deformation neighborhood basis of B in Y. □

Applying corollary 4.6 and lemma 5.7 we obtain the main result of this section.

Theorem 5.8 *Let B be a compact metric G-space. If B is absolutely neighborhood G-contractible then B is a G-ANR divisor.*

6 Homotopy characterization of absolute neighborhood G-contractibility

In this section we shall prove that the canonical projection $p : Y \to Y/B$, where $Y \in G\text{-}ANR(B)$, is a G-homotopy equivalence when B is compact and absolutely neighborhood G-contractible. Case when B is G-contractible the same conclusion remains valid too.

Theorem 6.1 *Let B be a compact metric G-space. Then are equivalent*

(a) B is absolutely neighborhood G-contractible.

(b) For every $Y \in G\text{-}ANR(B)$, the canonical projection $p : Y \to Y/B$ is a G-homotopy equivalence.

(c) For every $Y \in G\text{-}ANR(B)$, p has a left G-homotopy inverse.

Proof. $(a) \Rightarrow (b)$. Let $Y \in G\text{-}ANR(B)$ where B is absolutely neighborhood G-contractible. By theorem 5.8, $Y/B \in G\text{-}ANE$. Hence, there exits an invariant neighborhood U of $p(B)$ in Y/B and a G-contraction j_t, from U to $p(B)$ in Y/B, defined by

$$j_0 = i, j_1 = c, j_t = r, t \in (0, 1)$$

where i is the inclusion of U into Y/B, c is the constant G-map $c(u) = p(B)$ for every u in U, and $r : U \to Y/B$ is a G-map such that extends the inclusion of $p(B)$ into Y/B to U.

Thus, $j_t(p(B)) = p(B)$ for all $t \in I$. Applying theorem 2.4, we obtain a G-homotopy $J_t : Y/B \to Y/B$ such that extends j_t. Besides J_0 is the identity over Y/B.

Since J_1 extends simetrically j_1, we have

$$J_1(U) = p(B). \tag{1}$$

Due to that $p^{-1}(U)$ is open in Y, $p^{-1}(U)$ is a G-ANR by theorem 2.2, and therefore $p^{-1}(U) \in G\text{-}ANR(B)$. Since B is absolutely neighborhood G-contractible by theorem 5.6, there exists an invariant neighborhood V of B in $p^{-1}(U)$ such that the every metric G-pair (X, A), each G-map $f : A \to \overline{V}$ has an equivariant extension $F : X \to p^{-1}(U)$. Let k_t be a G-contraction of B to a point b_0 into V. Define the G-map

$$f : (\overline{V} \times \{0\}) \cup (B \times I) \cup (\overline{V} \times \{1\}) \to \overline{V}$$

by

$$f(y,t)=\begin{cases} y, & \textit{if } y\in\overline{V} \textit{ and } t=0,\\ k_t(y), & \textit{if } y\in B \textit{ and } 0\le t\le 1,\\ b_0, & \textit{if } y\in\overline{V} \textit{ and } t=1.\end{cases}$$

Since the image of f is contained of $\overline{V}$, by theorem 5.6, f equivariantly can be extended to a G-map $F:\overline{V}\times I\to p^{-1}(U)$. By theorem 2.4, we obtain a G-homotopy $K_t:Y\to Y$ such that $K_t(y)=F(y,t)$ for all $y\in\overline{V}$ and $0\le t\le 1$, and such that K_0 is the identity over Y. It is clear that K_t extends k_t; consequently,

$$K_1(B)=b_0 \tag{2}$$

and

$$K_t(B)\subset p^{-1}(U). \tag{3}$$

Let i and j be identity G-maps of Y and Y/B, respectively. Let $\varphi=K_1p^{-1}:Y/B\to Y$. By (2), φ is well-defined and due to that p is an identification, φ is continuous and clearly equivariant. We shall show that φ is an inverse G-homotopy of p.

In agreement (1) y (3), we can to see that $J_1pK_tp^{-1}$ is a well-defined G-homotopy between G-maps $J_1pK_0p^{-1}$ and $J_1pK_1p^{-1}$ over Y/B. Then we can write

$$j=J_0\overset{G}{\sim}J_1=J_1pK_0p^{-1}\overset{G}{\sim}J_1pK_1p^{-1}=J_1p\varphi\overset{G}{\sim}p\varphi:Y/B\to Y/B$$

At the same way

$$i=K_0\overset{G}{\sim}K_1=K_1p^{-1}p=\varphi p:Y\to Y$$

Then we conclude that p is a G-homotopic equivalence with G-homotopy inverse φ.

$(b)\Rightarrow(c)$ It is trivial.

$(c)\Rightarrow(a)$ Let $Y\in G\text{-}ANR(B)$ and let $q:Y/B\to Y$ be a left G-homotopy inverse of p. Then $qp\overset{G}{\sim}id_Y$, where id_Y is the identity on Y. However $qp(B)$ is a single point. Hence B is a G-contractible into Y and by theorem 5.4 (b), B is absolutely neighborhood G-contractible. □

The following theorem generalizes the equivalence of statements (b) and (c) of theorem 5.4.

Theorem 6.2 *Let B be a metric G-space. Thus, B is absolutely neighborhood G-contractible if and only if every G-map of B into a G-space $Y\in G\text{-}ANR$ is G-nullhomotopic.*

Proof. Let $f : B \to Y$ a G-map. Let $X \in G\text{-}ANR(B)$. Since $Y \in G\text{-}ANR$, f has an equivariant neighborhood extension $F : U \to Y$. Since B is absolutely neighborhood G-contractible, applying theorem 5.4 , we have that the inclusion $i : B \to U$ is G-nullhhomotopic. Thus $f = F \circ i$ is G-nullhomotopic.

Conversely, let U be an invariant neighborhood of B in $Y \in G\text{-}ANR(B)$. Hence U is a $G\text{-}ANR(B)$ and by hypothesis, the inclusion $i : B \to U$ is G-nullhomotopic and therefore, B is absolutely neighborhood G-contractible. □

From the previous theorem immediately follows

Corollary 6.3 *Let B be a metric G-space. If B is G-homotopically dominated by an absolutely neighborhood G-contractible G-space A, then B is absolutely neighborhood G-contractible.*

Proof. Let $f : B \to A$ be a G-map such that there exists a G-map $h : A \to B$ satisfying $hf \overset{G}{\sim} id_B$, where id_B is the identity of B. Let $l : B \to Y$ an arbitrary G-map, where $Y \in G\text{-}ANR$. We shall prove that l is G-nullhomotopic, and by previous theorem the proof will be completed. Let $k = l \circ h : A \to Y$; then by (b) of theorem 5.4, there exists a G-homotopy $k_t : A \to Y$ such that $k_0 = k$ and k_1 is a constant G-map. Now, define $L_t : B \to Y$ by $L_t = k_t \circ f$. Then, we have

$$L_0 = k_0 \circ f = k \circ f = l \circ h \circ f \overset{G}{\sim} l$$

On the other hand $L_1 = k_1 \circ f$ is a constant map and $L_0 \overset{G}{\sim} L_1$. Hence l is nullhomotopic. □

Remark 6.4 *Corollary say us that the absolute neighborhood G-contractibility is an invariant of the G-homotopy type between metric G-spaces. From here, every invariant retract of an absolutely neighborhood G-contractible G-space, is too.*

Theorem 6.5 *Let B be a metric G-space. If B is G-homotopically dominated by a G-ANR divisor A then B is a G-ANR divisor.*

Proof. Let $X \in G\text{-}ANR(A)$, $Y \in G\text{-}ANR(B)$, and $p : X \to X/A$, $q : Y \to Y/B$ the canonical projections. Let $f : B \to A$ be a G-map such that there exists a G-map $h : A \to B$ satisfying $hf \overset{G}{\sim} id_B$, where id_B is the identity of B. Denote by α_t the G-homotopy between hf and id_B. Since Y is a G-ANR then there exists an equivariant extension map $\phi : N \to Y$ of h where N is an invariant neighborhood of A in X. Since N is an invariant open in X, then by theorem 2.2, N is a G-ANR and consequently N/A is a G-ANE due to that A is a G-ANR divisor. By lemma 4.2 and the fact of N/A is an invariant neighborhood of $p(A)$, there exists a strong G-contraction $h_t : W \to N/A$

such that $h_t(p(A)) = p(A)$ and W is an invariant neighborhood of $p(A)$ in X/A. Since $p^{-1}(W)$ is an invariant open in X, $p^{-1}(W)$ is a G-ANR. Thus, there exists a G-extension $\psi : U \to p^{-1}(W)$ of f, where U is an invariant neighborhood of B in Y.

Define a G-map $\lambda : (U \times \{0\}) \cup (B \times I) \cup (U \times \{1\}) \to Y$ by

$$\lambda(u,t) = \begin{cases} u, & \textit{if } u \in U \textit{ and } t = 0, \\ \alpha_t(u), & \textit{if } u \in B \textit{ and } 0 \leq t \leq 1, \\ \phi \circ \psi(u), & \textit{if } u \in U \textit{ and } t = 1. \end{cases}$$

Since $Y \in G$-ANR, we can extend λ to a G-map $J : E \to Y$, where E is an invariant neighborhood of $(U \times \{0\}) \cup (B \times I) \cup (U \times \{1\})$ in $U \times I$. Let V be an invariant neighborhood of B in U such that $V \times I \subset E$ and $J(V \times I) \subset U$. Hence the restriction $J|_{V\times I}$ define a G-homotopy $j_t : V \to U$ such that j_0 is the identity G-map on V, $j_1 = \phi \circ \psi|_V$ and $j_t(B) \subset B$ for all t. Finally, we define a G-map $k : q(V) \times I \to Y/B$ by

$$k_t(x) = \begin{cases} qj_{2t}q^{-1}(x), & \textit{if } x \in q(V) \textit{ and } 0 \leq t \leq \frac{1}{2}, \\ q\varphi p^{-1}h_{2t-1}p\psi q^{-1}(x), & \textit{if } x \in q(V) \textit{ and } \frac{1}{2} \leq t \leq 1. \end{cases}$$

It is easy to check that k strongly G-deforms $q(V)$ in $q(B)$ and the proof is complete. □

Remark 6.6 *The previous theorem state that the property of to be a G-ANR divisor is an invariant of the G-homotopy type and consequently every retract of a G-ANR divisor is a G-ANR divisor.*

7 Quotients and unions of G-ANR divisors and absolutely neighborhood G-contractible spaces

Let A a compact G-ANR contained in a metric G-space B. By theorem 2.5, when B is a G-ANR, B/A is a G-ANR. But if B/A is a G-ANR it not implies that B is a G-ANR. We will shows that in this case, B is at least a G-ANR divisor (see remark 3.6).

Theorem 7.1 *Let (B, A) be a metric G-par, where A is a compact G-ANR divisor. Then B is a G-ANR divisor if and only if B/A is an G-ANR divisor.*

Proof. Suppose that B is a G-ANR divisor and let $Y \in G$-$ANR(B)$. Then Y/A is a G-ANR because A is a compact G-ANR divisor. It is clear that $Y/A \in G$-$ANR(B/A)$. Y/B is G-homeomorphic to $(Y/A)/(B/A)$. Besides Y/B is a G-ANE and Y/A is a G-ANR. Therefore, B/A is a G-ANR divisor.

Now, let B/A is a G-ANR divisor. Then $Y/B \cong (Y/A)/(B/A)$ is a G-ANE. Thus, B is a G-ANR divisor. □

In case when A is a G-AR compact subset of B in the previous theorem, the equivalence is not true always. However, the first implication is true while the second implies only that if B/A is a G-AR then B is a G-ANR.

If we change the condition G-ANR by absolutely neighborhood G-contractible in the theorem 7.1, then the affirmation is valid too. For to prove this we need before the following lemma.

Lemma 7.2 *Let (B, A) a G-pair such that both A and B are absolutely neighborhood G-contractible. Let $Y \in G$-$ANR(B)$ and let U be an invariant neighborhood of A in Y. Then B is G-deformable into U under a G-deformation that leaves A pointwise fixed.*

Proof. Let (X, C) a metric G-pair. Then by theorem 5.6, there exists an invariant neighborhood V of B in Y such that each G-map $f : C \to \overline{V}$ has an invariant extension $F : X \to Y$. Similarly there exists an invariant neighborhood W of A in $U \cap V$ such that each G-map $H : C \to \overline{W}$ has an invariant extension $H : X \to U \cap V$. Fix a point $a \in A$ and define a G-map $h^* : A \cup (B \backslash W) \to W$ by

$$h^*(x) = \begin{cases} x, & \textit{if } x \in A, \\ a, & \textit{if } x \in (B \backslash W). \end{cases}$$

Hence h^* has an equivariant extension $H^* : B \to U \cap V$. Now define a G-map $f^* : (B \times \{0\}) \cup (A \times I) \cup (B \times \{1\}) \to V$ by

$$f^*(x,t) = \begin{cases} x, & \textit{if } x \in B \textit{ and } t = 0, \\ x, & \textit{if } x \in A \textit{ and } 0 \le t \le 1, \\ H^*(x), & \textit{if } x \in B \textit{ and } t = 1. \end{cases}$$

Then there exits a G-extension $F^* : B \times I \to Y$ for f^*. It is easy to check that F^* is the desires G-deformation. □

Theorem 7.3 *Let (B, A) be a metric G-pair, where A is a compact absolutely neighborhood G-contractible. Then B is absolutely neighborhood G-contractible if and only if B/A is an absolutely neighborhood G-contractible.*

Proof. Let $Y \in G$-$ANR(B)$ and let $p : Y \to Y/A$ the canonical projection. Suppose that B is absolutely neighborhood G-contractible. In agreement with theorem 5.4 we shall shows that B/A is G-contractible in an arbitrary invariant neighborhood U of B/A in Y/A. Y/A is a G-ANR and since U is open in Y/A then U is a G-ANR. Hence there exists an invariant neighborhood V of $p(A)$

G-contractible in U. By lemma 7.2 there exists a G-deformation $k_t : B \to p^{-1}(U)$ leaving A pointwise fixed and such that $k_1(B) \subset p^{-1}(V)$. The G-homotopy $pk_t(p|_B)^{-1} : B/A \to U$ equivariantly deforms B/A into V, which is G-contractible in U. So, B/A is G-contractible in U and if follows that B/A is absolutely neighborhood G-contractible.

Now suppose that B/A is absolutely neighborhood G-contractible. Since Y/A is G-ANR and hence B/A is G-contractible in Y/A under a G-deformation k_t. By theorem 6.1, p has a G-homotopy inverse $q : Y/A \to Y$. Let $i : B \to Y$ the inclusion. Then we have $i \overset{G}{\sim} qp|_B = qk_0p|_B \overset{G}{\sim} qk_1p|_B$. Since k_1 is constant, i is nullhomotopic G-map and by theorem 5.4, B is absolutely neighborhood G-contractible. □

The fact of build up G-ANR starting from others G-ANR is possible if the last G-ANR have some property (see [4], Theorem 5.1). However, is the union of G-ANR divisors a G-ANR divisor again? First we shall show that the disjoint union of G-ANR divisors is G-ANR divisor and later we generalize this fact.

Lemma 7.4 *Let A_1 and A_2 be a compact G-ANR divisors such that $A_1 \cap A_2 = \emptyset$. Then $A_1 \cup A_2$ is a G-ANR divisor.*

Proof. Let $Y \in G\text{-}ANR(A_1 \cup A_2)$. Then Y/A_1 is a G-ANR. Let $p : Y \to Y/A_1$ the canonical projection. It is clear that $p|_{A_2}$ is a G-homeomorphism. Hence $p(A_2)$ is a G-ANR divisor; so $\frac{(Y/A_1)}{p(A_2)}$ is a G-ANR. Let $q : Y/A_1 \to \frac{(Y/A_1)}{p(A_2)}$ the canonical projection. It is easy to see that $qp(A_1)$ and $qp(A_2)$ are singletons. Thus, they and their union are G-ANR. By theorem 2.5, $\frac{(Y/A_1)}{p(A_2)}/(qp(A_1) \cup qp(A_2))$ is a G-ANR, but it is homeomorphic to $Y/(A_1 \cup A_2)$ and we conclude that $A_1 \cup A_2$ is a G-ANR divisor. □

Theorem 7.5 *Let B compact metric G-space. Let $\{B_i\}_{i=1}^n$ be an invariant closed cover of B. If for each finite sub-collection $\{A_{i_j}\}$ of $\{B_i\}$, the intersection $\bigcap A_{i_j}$ is a G-ANR divisor (or empty), then B is a G-ANR divisor.*

Proof. We shall proceed by induction. It is trivial that the theorem is true for $n = 1$. Suppose that it is true for $n = k$ and we shall prove that if verifies for $n = k + 1$. Let $C_i = B_i \cap B_{k+1}$, for $i = 1, 2, ..., k$. By hypothesis, each C_i is a G-ANR divisor (or empty). Let $D = \cup_{i=1}^k B_i$ and $E = \cup_{i=1}^k C_i$; by induction hypothesis D and E are G-ANR divisors (or empty). Then we have two cases:

i) If $E \neq \emptyset$, then by theorem 7.1, B_{k+1}/E is a G-ANR divisor. But B_{k+1}/E is G-homeomorphic to B/D. Again applying theorem 7.1 B is a G-ANR divisor.

ii) If $E = \emptyset$, then B is the disjoint union of B_{k+1} and D. By Lemma 7.4, the proof is complete. □

From theorems 5.8 and 7.5 follows

Corollary 7.6 *Let B compact metric G-space. Let $\{B_i\}_{i=1}^n$ be an invariant closed cover of B. If for each finite sub-collection $\{A_{i_j}\}$ of $\{B_i\}$, the intersection $\bigcap A_{i_j}$ is absolutely neighborhood G-contractible (or empty), then B is a G-ANR divisor.*

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