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Relation connecting Zagreb co-indices on Three graph Operators

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Abstract

The aim of this paper is to show Zagreb co-indices, an important invariant of a graph changes with several graph operators on the subdivision graphs of some connected graphs. Also, derived expressions for the relationship connecting Zagreb co-indices on three graph operators.

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1 Introduction and Terminology

A topological index is a numerical invariant of a chemical graph. Two most commonly used graph invariants are the first and second Zagreb indices. They have played significant role in the study of Topological indices. Due to their chemical relevance, they have been subject of numerous papers in the chemical literature. This fascinating area has been enriched by its formidable links to combinatorics, Number theory and some other branches; all deeply cross-fertilize each other holding great promise for all them. Recently, P. S. Ranjini and V. Lokesha derived an expression for Zagreb indices and co-indices of the subdivision graph of tadpole and wheel graphs ([10], [11]). They have also derived a relation connecting the Zagreb indices on three graph operators [9]. The aim of this paper is to produce an expression for the relation connecting Zagreb co-indices on three graph operators $S(G), R(G)$ and $Q(G)$.

Let G be a connected graph with vertex set and edge sets $V(G)$ and $E(G)$ respectively. For every vertex $u \in V(G)$, the edge connecting u and v is denoted by uv and $d(u)$ denotes the degree of u in G.

The first and the second zagreb indices [1] are defined as follows.

$$M_1(G) = \sum [d(u)^2] \tag{1}$$

$$M_2(G) = \sum [d(u)d(v)], \ uv \in E(G). \tag{2}$$

We refer the reader to [3] for the proof of this fact.

The first and the second Zagreb co-indices were first introduced by Doslic. They are defined as follows. For uv not belongs to the edge set $E(G)$,

$$\overline{M_1(G)} = \sum [d(u) + d(v)]$$

$$\overline{M_2(G)} = \sum [d(u)d(v)].$$

The *subdivision graph* [10] $S(G)$ is the graph obtained from G by replacing each of its edge by a path of length 2 or equivalently, by inserting an additional vertex into each edge of G [13]. The operator $R(G)$ is the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the edge corresponding to it. The $T_{n,k}$ *tadpole graph* [14] is the graph obtained by joining a cycle graph C_n to a path of length k. The *wheel graph* W_{n+1} [7] is defined as the graph $K_1 + C_n$, where K_1 is the singleton graph and C_n is the cycle graph [[5], [6],[12], [8]].

The outline of the paper is as follows: Introduction and terminologies are described in the first section. In section 2, we calculated the Zagreb co-indices of $T_{n,k}$ a relationship connecting Zagreb co-indices on three graph operators are derived. The last section devoted Zagreb co-indices of W_{n+1} to same class of graph operators considered in preceding section.

2 Relation connecting Zagreb co-indices on S(G), R(G) and Q(G) for the Tadpole Graph

In this section, we derive an expression for the relationship connecting the *Zagreb co-indices* on three graph operators $S(G), R(G)$ and $Q(G)$ for *tadpole graph* $T_{n,k}$ and *wheel graph* W_{n+1}.

Theorem 2.1 *For the subdivision graph of a tadpole graph, the first Zagreb co-index is*

$$\overline{M_1(S(T_{n,k}))} = 4\overline{M_1(T_{n,k})} + 6(2n + 2k + 1).$$

Proof: The tadpole graph $T_{n,k}$ contains $n + k - 2$ vertices of degree 2, one vertex of degree 3 and a pendent vertex. The subdivision graph $S(T_{n,k})$ contains $(n + k)$ additional vertices of degree 2. In $T_{n,k}$, let v_1 be the vertex of degree 3 and $v_{1'}$ and $v_{2'}$ be the neighbours of v_1 in the cycle C_n and v_j be the neighbour of v_1 in the path P_k. Let v_l be the pendent vertex in $T_{n,k}$.
We calculate $\sum[d(u) + d(v)]$,

1. Among the vertices in C_n
2. From cycle C_n to the path P_k
3. Among the vertices in the path P_k

In C_n, $v_{1'}$ and $v_{2'}$ makes the sum $\sum[d(u)+d(v)]$ with $n-3$ vertices. Remaining $n - 3$ vertices in C_n makes the sum 4 with $n - 4$ vertices and make the sum 5 with v_1. Also v_1 make the sum 5 with $n - 3$ vertices in C_n. Hence in C_n $\sum[d(u) + d(v)]$ is $4n^2 - 10n - 6$.

Since one edge is shared between a pair of vertices, $\sum[d(u) + d(v)]$ in C_n is

$$2n^2 - 5n - 3 \tag{3}$$

From cycle C_n to path P_k, all the $n - 1$ vertices other than v_1 in C_n makes the sum 3 with respect to the vertex v_l. Also all the $n - 1$ vertices makes the sum 4 with $k - 1$ vertices in the path other than v_1.

So from C_n to the path P_k, $\sum[d(u) + d(v)]$ is

$$(n - 1)(4k - 1) \tag{4}$$

In the path $T(P_k)$, the vertex v_1 makes the sum 5 with $k - 2$ vertices in the path and makes 4 with the vertex v_l. The vertex v_j in the path makes the sum 4 with the remaining $k - 4$ vertices in the path and so on. Hence in the path $\sum[d(u) + d(v)]$ is

$$2k^2 - 2k. \tag{5}$$

Adding equations 3, 4 and 5

$$\overline{M_1(T_{n,k})} = 2n^2 - 6n + 4nk + 2k^2 - 6k - 2.$$

$$\overline{M_1(T_{n,k})} = 2(n^2 + k^2) - 6(n + k) + 2(2nk - 1)$$
$$\overline{M_1(S(T_{n,k}))} = 8(n^2 + k^2) - 12(n + k) + 2(8nk - 1)$$

Hence

$$\overline{M_1(S(T_{n,k}))} = 4\overline{(M_1(T_{n,k})} + 6(2n + 2k + 1).$$

Theorem 2.2 *The second Zagreb co-index is*

$$\overline{M_2(S(T_{n,k}))} = \overline{M_1(S(T_{n,k}))} - 3.$$

Proof: All the $2n - 1$ vertices which make the sum 3 in $\overline{M_1(S(T_{n,k}))}$, make the product 2 in $\overline{M_2(S(T_{n,k}))}$. All the $(2n - 3)$ vertices which make the sum 5 in $\overline{M_1(S(T_{n,k}))}$ make the product 6 in $\overline{M_2(S(T_{n,k}))}$. Also $\overline{M_1(S(T_{n,k}))}$ contains a pair of vertices which makes the sum 4 and $\overline{M_2(S(T_{n,k}))}$ also contains an additional pair of vertices which makes the product 3. Hence

$$\overline{M_2(S(T_{n,k}))} = \overline{M_1(S(T_{n,k}))} - 3.$$

Theorem 2.3 *For the Tadpole graph $T_{n,k}$,*

$$\overline{M_1(R(T_{n,k}))} = \overline{M_1(S(T_{n,k}))} + 2\overline{M_1(T_{n,k})} - 2(n + k + 1).$$

Proof: The Vertices which are of degree l in $S(T_{n,k})$ are of degree $2l$ in $R(T_{n,k})$. All the subdivision vertices are of the same degree in both $S(T_{n,k})$ and in $R(T_{n,k})$.

In the cycle $R(C_n)$, the vertices which are adjacent to v_1 makes the sum of degrees 8 with remaining $n - 3$ vertices in the cycle and the remaining $n - 3$ vertices make the sum 8 with $n - 4$ vertices in the cycle. Also v_1 makes the sum 10 with the $n - 3$ vertices. All the n subdivision vertices make the sum 4 with the remaining $n - 1$ subdivision vertices. The vertex v_1 makes the sum 8 with $n - 2$ subdivision vertices. The $n - 1$ vertices other than v_1 make the sum 6 with the $n - 2$ sub division vertices.

Hence in $R(C_n)$, $\sum[d(u) + d(v)]$ is

$$12n^2 - 22n - 10 \tag{6}$$

To calculate $\sum[d(u) + d(v)]$ from $R(C_n)$ to the path, all the $n - 1$ vertices in the cycle other than v_1 make the sum 6 with v_l and k subdivision vertices in the path. All the n subdivision vertices in the cycle make the sum 4 with v_l

and k subdivision vertices in the path. Also all n subdivision vertices in C_n make the sum 6 with $k-1$ vertices in the path.

Hence from cycle to path $\sum[d(u)+d(v)]$ is

$$24nk - 4n - 14k + 2 \tag{7}$$

In the path $R(P_k)$, vertex v_1 makes the sum 8 with $k-1$ subdivision vertices in the path as well as with v_l. Also v_1 make the sum 10 with $k-2$ vertices in the path.The subdivision vertex v_j in the path make the sum 4 with the remaining $k-1$ subdivision vertices as well as with v_l. It also makes the sum 6 with $k-2$ vertices in the path. The neighbours of v_j in the path make the sum 8 with $k-3$ vertices and 6 with $k-2$ vertices and so on.

Hence in the path $\sum[d(u)+d(v)]$ is

$$12k^2 - 12k \tag{8}$$

Adding equations 6, 7 and 8,

$$\overline{M_1(R(T_{n,k}))} = 12(n^2+k^2) - 26(n+k) + 8(3nk-1).$$

Hence in $R(T_{n,k})$

$$\overline{M_1(R(T_{n,k}))} = \overline{M_1(S(T_{n,k}))} + 2\overline{M_1(T_{n,k})} - 2(n+k+1).$$

Theorem 2.4 *For the Tadpole graph $T_{n,k}$,*

$$\overline{M_2(R(T_{n,k}))} = \begin{cases} 2\overline{M_2(S(T_{n,k}))} + \overline{M_2(T_{n,k})} - 2(6n+6k+5), & \text{when } k = 1; \\ 2\overline{M_2(S(T_{n,k}))} + \overline{M_2(T_{n,k})} - 4(3n+3k+4), & \text{when } k > 1. \end{cases}$$

Proof:For the Tadpole graph $T_{n,k}$, $\sum[d(u).d(v)]$ with respect to cycle C_n is

$$2n^2 - 4n - 6 \tag{9}$$

$\sum[d(u).d(v)]$ with respect to cycle C_n to path P_k is

$$4nk - 2n - 4k + 2 \tag{10}$$

When $k=1$,$\sum[d(u).d(v)]$ among the vertices in the path P_k is

$$2k^2 - 2k \tag{11}$$

When $k>1$, $\sum[d(u).d(v)]$ among the vertices in the path P_k is

$$2k^2 - 2k - 1 \tag{12}$$

Adding the above equations,

$$\overline{M_2(T_{n,k})} = \begin{cases} 2(n^2+k^2) - 6(n+k) + 4(nk-1), & \textit{when } k{=}1; \\ 2(n^2+k^2) - 6(n+k) + 4(nk-1) - 1, & \textit{when } k>1. \end{cases}$$

In the cycle $R(C_n)$, two vertices which are adjacent to v_1 make the product 16 with $n-3$ vertices and the remaining $n-3$ vertices makes 16 with $n-4$ vertices. Since one edge is shared between two vertices, it makes the product $16(n-3)+8(n-3)(n-4)$. Also v_1 make the product 24 with $n-3$ vertices. Each of the $n-1$ vertices other than v_1 makes the product 8 with $n-2$ subdivision vertices and v_1 make 12 with $n-2$ subdivision vertices. Each of the n subdivision vertices makes the product 4 with $n-1$ subdivision vertices. Since one edge is shared between two vertices, it makes the product $2n(n-1)$.

So in $R(C_n)$,

$$\sum[d(u).d(v)] = 18n^2 - 30n - 32 \tag{13}$$

From cycle C_n to the path P_k, $n-1$ vertices other than v_1 make the product 16 with $k-1$ vertices in the path and the same vertices make the product of degrees 8 with k subdivision vertices and also with v_l. The n subdivision vertices in C_n makes the product 8 with the $k-1$ vertices in the path and makes 4 with k subdivision vertices in the path as well as with v_l.

Hence $\sum[d(u).d(v)]$ from cycle to path is

$$(12n-8)(3k-1) \tag{14}$$

To calculate $\sum[d(u).d(v)]$ among the vertices in the path P_k.
Case 1: When $k=1$,

$$\sum[d(u).d(v)] = 18k^2 - 18k \tag{15}$$

Adding equations 13, 14 and 15,

$$\overline{M_2(R(T_{n,k}))} = 18(n^2+k^2) - 42(n+k) + 4(9nk-6)$$

Hence for $k=1$,

$$\overline{M_2(R(T_{n,k}))} = 2\overline{M_2(S(T_{n,k}))} + \overline{M_2(T_{n,k})} - 2(6n+6k+5).$$

Case 2: The vertex v_1 makes the product 24 with $k-2$ vertices and make the product 12 with $k-1$ subdivision vertices as well as with v_l. The vertex v_j in the path make 4 with the remaining $k-1$ subdivision vertices as well as with v_l. It also make 8 with $k-2$ vertices in the path. The remaining vertices in the path makes the product 16 with $k-3$ vertices and 8 with $k-2$ vertices and so on.

Hence $\sum[d(u).d(v)]$ in the path is

$$18k^2 - 18k - 4, \ \textit{when } k > 1 \tag{16}$$

Adding equations 13, 14 and 16,

$$\overline{M_2(R(T_{n,k}))} = 18(n^2 + k^2) - 42(n + k) + 4(9nk - 7), \ \textit{when } k > 1.$$

$$\overline{M_2(R(T_{n,k}))} = 2\overline{M_2(S(T_{n,k}))} + \overline{M_2(T_{n,k})} - 4(3n + 3k + 4), \ \textit{when } k > 1.$$

Theorem 2.5 *For the Tadpole graph $T_{n,k}$*

$$\overline{M_1(Q(T_{n,k}))} = \begin{cases} \overline{M_1(R(T_{n,k}))} + 2(2n + 2k - 7), & \textit{when } k = 1; \\ \overline{M_1(R(T_{n,k}))} + 2(2n + 2k - 8), & \textit{when } k > 1. \end{cases}$$

Proof: When $k = 1$, the graph $Q(T_{n,k})$ contains the subgraph $T_{n,k}$. The $n + k - 2$ subdivision vertices of degree 2 in $S(T_{n,k})$ are of double the degree in $Q(T_{n,k})$ and only 2 vertices of degree 5. For $k > 2$, the $n + k - 4$ subdivision vertices of degree 2 in $S(T_{n,k})$ is of degree 4 in $Q(T_{n,k})$ and only 3 vertices of degree 5 and one vertex of degree 3.

$\sum[d(u) + d(v)]$ **with respect to the cycle** C_n **of** $Q(T_{n,k})$**.**

Each $n - 1$ vertices of degree 2 make the sum 4 with $n - 2$ vertices and make 5 with v_1. The vertices adjacent to subdivision vertices of degree 5 make the sum 6 with $n - 3$ subdivision vertices and make 7 with the two subdivision vertices of degree 5. The remaining $n - 3$ vertices make 6 with $n - 4$ subdivision vertices, make 7 with subdivision vertices of degree 5. Subdivision vertices of degree 4 which are adjacent to subdivision vertices of degree 5 make the sum 8 with $n - 4$ subdivision vertices and these $n - 4$ subdivision vertices makes the sum 8 with $n - 5$ subdivision vertices. The two Subdivision vertices of degree 5 in the cycle make the sum 9 with $n - 3$ subdivision vertices. The vertex v_1 of degree 3 in the cycle makes the sum 7with $n - 2$ subdivision vertices. Hence $\sum[d(u) + d(v)]$ with respect to the cycle C_n is

$$12n^2 - 20n - 13 \tag{17}$$

$\sum[d(u) + d(v)]$ **with respect to the cycle** C_n **to the path** P_k**.**

When $k = 1$, the path of $Q(T_{n,k})$ contains a vertex of degree 1 and a subdivision vertex of degree 4. The $n - 1$ vertices in the cycle makes the sum 3 with the pendent vertex in the path and makes the sum of degrees 6 with the subdivision vertex of degree 4. The $n - 2$ subdivision vertices in the cycle makes the sum 5 with the pendent vertex and makes 8 with subdivision vertex of degree 4 in the path. The two subdivision vertices of degree 5 in the cycle make the sum 6 with the pendent vertex.

Hence $\sum[d(u) + d(v)]$ with respect to the cycle to the path is

$$22n - 23 \tag{18}$$

When $k \geq 2$, the $n-1$ vertices of degree 2 in the cycle makes the sum 3 with the pendent vertex in the path, makes 4 with $k-1$ vertices, makes 5 with the vertex of degree 3 in the path, makes the sum 7 with the subdivision vertex of degree 5 and makes the sum 6 with $k-2$ subdivision vertices. The two subdivision vertices of degree 5 in the cycle makes the sum 6 with the pendent vertex, makes 7 with $k-1$ vertices, makes 8 with subdivision vertex of degree 3 in the path and makes the sum 9 with $k-2$ subdivision vertices in the path. The $n-2$ subdivision vertices in the cycle makes the sum 5 with the pendent vertex, makes 6 with $k-1$ vertices in the path, makes 7 with vertex of degree 3 and makes the sum 9 with subdivision vertex of degree 5 in the path.

Hence from cycle to path,

$$\sum[d(u)+d(v)] = 24nk - 6k - 19, \ \textit{when } k > 1 \tag{19}$$

$\sum[d(u)+d(v)]$ **with respect to the path P_k.**

In the path P_k, let v_1 be the vertex of degree 3, v_j be the vertex of degree 5, $v_{j'}$ be the neighbour of v_j, $v_{j''}$ is the subdivision vertex which is adjacent to $v_{j'}$ and so on. The vertex v_1 makes the sum 5 with $k-1$ vertices and with $k-2$ subdivision vertices in the path. It also makes the sum 6 with the subdivision vertices of degree 3 and makes 4 with the pendent vertex. The vertex v_j in the path makes the sum 7 with $k-2$ vertices and makes 9 with $k-3$ subdivision vertices. It also makes the sum 8 with subdivision vertices of degree 3 and makes 6 with the pendent vertex. The vertex $v_{j'}$ makes the sum 4 with $k-2$ vertices, 6 with $k-3$ subdivision vertices, 5 with subdivision vertex of degree 3 and makes the sum 3 with the pendent vertex. The next vertex in the path makes the sum 4 with $k-3$ vertices and so on. The vertex $v_{j''}$ makes the sum 6 with $k-3$ vertices, 8 with $k-4$ vertices, 7 with subdivision vertex of degree 3 and 5 with the pendent vertex. The next subdivision vertex in the path makes the sum 6 with $k-4$ vertices and so on. Hence in the path,

$$\sum[d(u)+d(v)] = 12k^2 - 16k + 8 \tag{20}$$

Adding the above equations,

$$\overline{M_1(Q(T_{n,k}))} = \begin{cases} \overline{M_1(R(T_{n,k}))} + 2(2n+2k-7), & \textit{when } k = 1; \\ \overline{M_1(R(T_{n,k}))} + 2(2n+2k-8), & \textit{when } k > 1. \end{cases}$$

Theorem 2.6 *For the Tadpole graph $T_{n,k}$*

$$\overline{M_2(Q(T_{n,k}))} = \begin{cases} \overline{M_2(R(T_{n,k}))} - (31-12n), & \textit{when } k = 1 \textit{ and } 2; \\ \overline{M_2(R(T_{n,k}))} - (4k^2 - 12n - 32k + 79), & \textit{when } k > 2. \end{cases}$$

Proof: $\sum[d(u).d(v)]$ **with respect to the vertices in the cycle C_n.**

Each of $n-1$ vertices other than v_1 make the product of degrees 4 with the remaining $n-2$ vertices and these $n-2$ vertices make the product 4 with $n-1$ vertices. The $n-1$ vertices makes the product 6 with v_1 and also v_1 makes the product 6 with each of the $n-1$ vertices. The vertices adjacent to the subdivision vertices of degree 5 makes the product 8 with $n-3$ vertices and make 10 with the two subdivision vertices of degree 5. Remaining $n-3$ vertices makes the product of degrees 8 with $n-4$ subdivision vertices and makes the product of degrees 10 with the two subdivision vertices of degree 5. The two subdivision vertices which are adjacent to the two subdivision vertices of degree 5 makes the product of degrees 16 with $n-4$ subdivision vertices and these $n-4$ subdivision vertices makes the product of degrees 16 with $n-5$ subdivision vertices. The two subdivision vertices of degree 5 make the product 20 with $n-3$ subdivision vertices and v_1 makes the product 12 with $n-2$ subdivision vertices.

Hence $\sum[d(u).d(v)]$ with respect to the vertices in the cycle C_n is

$$18n^2 - 24n - 42 \tag{21}$$

$\sum[d(u).d(v)]$ **from the vertices in the cycle** C_n **to the path** P_k**.**

Case 1: When $k = 1$, the $n-1$ vertices in the cycle make the product of degree 8 with v_j and 2 with v_l. Also, the subdivision vertices of degree 5 make the product 5 with v_l. Hence $\sum[d(u).d(v)]$ is

$$30n - 40, \ \textit{when } k\textit{=1} \tag{22}$$

Case 2: When $k > 1$**.**The $n-1$ vertices in the cycle make the product 10 with v_j, 2 with v_l, 4 with $k-1$ vertices in the path, 8 with $k-2$ subdivision vertices and makes the product 6 with subdivision vertices of degree 3 in the path.The two subdivision vertices of degree 5 in the cycle makes the product of degree 10 with $k-1$ vertices, 20 with $k-2$ subdivision vertices, 15 with vertex of degree 3 and 5 with pendent vertex. Remaining $n-2$ subdivision makes the product 20 with v_j, 8 with $k-1$ vertices, 16 with $k-2$ subdivision vertices, 12 with subdivision vertices of degree 3 and makes the product of degree 4 with v_l. Hence $\sum[d(u).d(v)]$ from cycle to the path is

$$36nk - 6n - 50, \ \textit{when } k > \textit{1} \tag{23}$$

$\sum[d(u).d(v)]$ **with respect to the path** P_k**.**

Case 1: When $k = 1$**.** In the path P_k, there is one vertex v_1 of degree 3, one vertex v_j of degree 4 and a pendent vertex v_l. Hence $\sum[d(u).d(v)]$ is

$$14k^2 - 10k - 1 \tag{24}$$

Case 2: When $k = 2$**.** The path P_k contains single vertices of degree 3, degree 2 and degree 1 and subdivision vertices of degree 4 and 3. Hence $\sum[d(u).d(v)]$ is

$$14k^2 - 10k - 11 \tag{25}$$

Case 3: When $k > 2$. In the path of $Q(T_{n,k})$, there are $k-1$ vertices of degree 2, a single vertex of degree 3 and 1. There is a single subdivision vertex of degree 5 and 3 and all the remaining $k-2$ subdivision vertices are of degree 4. The vertex of degree 3 makes the product 6 with $k-1$ vertices, makes 12 with $k-2$ subdivision vertices, makes 9 with vertex of degree 3 and makes the product 3 with the pendent vertex. The subdivision vertex of degree 5 makes the product 10 with $k-2$ vertices in the path, makes 20 with $k-3$ subdivisional vertices, makes 15 with vertex of degree 3 and makes the product 5 with the pendent vertex. The vertex which is adjacent to the subdivision vertex of degree 5 makes the product 4 with $k-2$ vertices, 8 with $k-3$ subdivision vertices, 6 with vertex of degree 3 and 2 with the pendent vertex. The next vertex of of degree 2 makes the product 4 with $k-3$ vertices, 8 with $k-4$ vertices, 6 with vertex of degree 3 and 2 with the pendent vertex and so on. The subdivision vertex adjacent to v_j makes the sum 8 with $k-3$ vertices in the path, 16 with $k-4$ subdivision vertices, 12 with vertex of degree 3 and makes the product of degree 4 with vertex of degree 1. The neighbors of this subdivision vertices make the product 8 with $k-4$ vertices, 16 with $k-5$ vertices and so on. Hence in the path $\sum[d(u).d(v)]$,

$$14k^2 - 10k - 12 \tag{26}$$

Adding the equations corresponding to the various cases,
For $k = 1$,

$$\overline{M_{2(}Q(T_{n,k}))} = 18n^2 - 30n + 36nk - 10k + 14k^2 - 83$$

$$= [18n^2 - 42n + 36nk - 42k + 14k^2 - 24] - [31 - 12n]$$

$$\overline{M_{2(}Q(T_{n,k}))} = \overline{M_2(R(T_{n,k}))} - [31 - 12n]$$

For $k = 2$,

$$\overline{M_{2(}Q(T_{n,k}))} = 18n^2 - 30n + 36nk - 10k + 14k^2 - 103$$

$$= [18n^2 - 42n + 36nk - 42k + 14k^2 - 24] - [31 - 12n]$$

$$\overline{M_{2(}Q(T_{n,k}))} = \overline{M_2(R(T_{n,k}))} - [31 - 12n]$$

For $k > 2$,

$$\overline{M_{2(}Q(T_{n,k}))} = 18n^2 - 30n + 36nk - 10k + 14k^2 - 104.$$

$$= [18n^2 - 42n + 36nk - 42k + 14k^2 - 24] - [4k^2 - 12n - 32k + 79]$$

$$= \overline{M_2(R(T_{n,k}))} - [4k^2 - 12n - 32k + 79]$$

Hence

$$\overline{M_{2(}Q(T_{n,k}))} = \begin{cases} \overline{M_2(R(T_{n,k}))} - [31 - 12n], & \textit{when } k = \textit{1 and 2}; \\ \overline{M_2(R(T_{n,k}))} - [4k^2 - 12n - 32k + 79], & \textit{when } k > \textit{2}. \end{cases}$$

3 Relation connecting Zagreb co-indices on S(G), R(G) and Q(G) for the Wheel Graph

A relation connecting Zagreb co-indices on S(G), R(G) and Q(G) for the Wheel graph In this section, we derive an expression for the relationship connecting the zagreb co-indices on three graph operators S(G), R(G)and Q(G) for the Wheel graph.

Theorem 3.1 *For the wheel graph W_{n+1},*

$$\overline{M_1(S(W_{n+1}))} = 7\overline{M_1(W_{n+1})} + 2n^2 + 46n.$$

Proof: In W_{n+1}, the hub of the wheel is of degree n and the n vertices are of degree 3. $n(n-3)/2$ pairs of vertices make the sum 6. Hence

$$\overline{M_1(W_{n+1})} = 3n^2 - 9n. \tag{27}$$

$S(W_{n+1})$ is formed by inserting a vertex in each edge of W_{n+1}. In $S(W_{n+1})$, $\frac{1}{2}n(n-1)$ pairs of vertices make the sum 4, n vertices make the sum $n+3$ with the hub of the wheel and $n(2n-3)$ pairs of vertices make the sum 5. Also $n(2n-1)$ pairs of subdivision vertices make the sum 4 and n subdivision vertices make the sum $n+2$ with hub of the wheel. Hence in $S(W_{n+1})$,

$$\overline{M_1(S(W_{n+1}))} = 23n^2 - 17n.$$

$$= 7(3n^2 - 9n) + (2n^2 + 46n).$$

$$= 7(\overline{M_1(W_{n+1}))} + (2n^2 + 46n).$$

Theorem 3.2 *The second Zagreb co-index of the subdivision graph of the Wheel graph is*

$$\overline{M_2(S(W_{n+1}))} = 6(\overline{M_2(W_{n+1}))} + \frac{1}{2}(5n^2 + 109n).$$

Proof: In W_{n+1}, $\frac{1}{2}n(n-1)$ pairs of vertices make the product 9. Hence

$$\overline{M_2(W_{n+1})} = \frac{1}{2}[9n^2 - 27n] \tag{28}$$

In $S(W_{n+1})$, n vertices make the product $3n$ with the hub of the wheel, make the product 9 with $(n-1)$ vertices and make 6 with $2n-3$ subdivision vertices. The hub of the wheel makes the product $3n$ with n vertices and $2n$ with subdivision vertices on the cycle C_n of the wheel. Subdivision vertices on the cycle C_n make the product 4 with the remaining subdivision vertices on the cycle and with n subdivision vertices on the spokes of the wheel. It also makes the product $2n$ with the hub of the wheel and makes 6 with $4(n-2)$ vertices. The n vertices on the spokes of the wheel make the product 4 with n vertices on the cycle C_n of the wheel and makes 4 with the remaining $(n-1)$ vertices on the spokes of the wheel. It also makes the product 6 with $(n-1)$ vertices on the cycle C_n. Hence in $S(W_{n+1})$,

$$\sum[d(u).d(v)] = \frac{1}{2}(118n^2 - 106n) \tag{29}$$

Since one edge is shared by a pair of vertices,

$$\overline{M_2(S(W_{n+1}))} = \frac{1}{2}(59n^2 - 53n).$$

$$= \frac{1}{2}(54n^2 - 162n) + \frac{1}{2}(5n^2 + 109n).$$

$$= 6(\overline{M_2(W_{n+1})}) + \frac{1}{2}(5n^2 + 109n).$$

Theorem 3.3 *For the Wheelgraph* W_{n+1},

$$\overline{M_1(R(W_{n+1}))} = \overline{M_1(S(W_{n+1}))} + 3(\overline{M_1(W_{n+1})}).$$

and

$$\overline{M_2(R(W_{n+1}))} = \overline{M_2(S(W_{n+1}))} + 5\overline{M_2(W_{n+1})} + 2n^2.$$

Proof: In $R(W_{n+1})$, n vertices are of degree 6, hub of the wheel is of degree $2n$ and all subdivision vertices are of degree 2. Hence $[d(u)+d(v)]$ and $\sum[d(u).d(v)]$ with respect to the hub of the wheel is respectively,

$$2n^2 + 2n \tag{30}$$

$$4n^2 \tag{31}$$

The sum of the degrees and the product of degrees with respect to all the n vertices of C_n, respectively given by

$$4n(7n - 15) \tag{32}$$

$$12n(5n - 12) \tag{33}$$

With respect to the n subdivision vertices on the spokes of the wheel, $\sum[d(u)+d(v)]$ and $\sum[d(u).d(v)]$is

$$4n(4n - 3) \tag{34}$$

$$4n(5n - 4) \tag{35}$$

The calculations with respect to n subdivision vertices on the edges of the cycle C_n of $R(W_{n+1})$ is,

$$18n(n-1) \tag{36}$$

$$\sum[d(u).d(v)] = 4n(6n-7). \tag{37}$$

Adding equations corresponding to the sum of degrees, we have

$$\sum[d(u)+d(v)] = 64n^2 - 88n.$$

Since one edge is shared by a pair of vertices,

$$\overline{M_1(R(W_{n+1}))} = 32n^2 - 44n.$$

$$\overline{M_1(R(W_{n+1}))} = \overline{M_1(S(W_{n+1}))} + 3\overline{(M_1(W_{n+1}))}.$$

Summing the equations corresponding to the product of degrees, we obtained,

$$\sum[d(u).d(v)] = 108n^2 - 188n.$$

Since one edge is shared between a pair of vertices,

$$\overline{M_2(R(W_{n+1}))} = 54n^2 - 94n$$

$$\overline{M_2(R(W_{n+1}))} = (52n^2 - 94n) + 2n^2$$

$$\overline{M_2(R(W_{n+1}))} = \frac{1}{2}(59n^2 - 53n) + \frac{5}{2}(9n^2 - 27n) + 2n^2$$

$$\overline{M_2(R(W_{n+1}))} = \overline{M_2(S(W_{n+1}))} + 5\overline{M_2(W_{n+1}))} + 2n^2.$$

Theorem 3.4 *For the wheel graph W_{n+1},*

$$\overline{M_1(Q(W_{n+1}))} = \overline{M_1(R(W_{n+1}))} + (2n^3 - 7n - 9)$$

and

$$\overline{M_2(Q(W_{n+1}))} = 2\overline{M_2(R(W_{n+1}))} + \overline{M_2(W_{n+1})} + 3n(n^2 - n + 7) - 18.$$

Proof: The $Q(W_{n+1})$ contains the subgraph W_{n+1}. Every vertex in $Q(W_{n+1})$ is of degree 3, the hub of the wheel is of degree n, the n subdivision vertices on the edge of the cycle C_n is of degree 6 and n subdivision vertices on the spokes of the wheel are of degree $n+3$.

From the hub of the wheel to all the other vertices, $\sum[d(u)+d(v)]$ and $\sum[d(u)d(v)]$ is

$$2n^2 + 9n \tag{38}$$

$$9n^2 \tag{39}$$

With respect to n vertices on the cycle C_n, the sum of the degrees and the product of the degrees are given respectively,

$$n^3 + 21n^2 - 39n \tag{40}$$

$$3n(n^2 + 12n - 24) \tag{41}$$

The calculations with respect to the n subdivision vertices on the spokes of the wheel is given by

$$2n^3 + 12n^2 - 6n - 18 \tag{42}$$

$$3n(n + 3)(3n - 5) \tag{43}$$

With respect to n subdivision vertices on the cycle C_n of $R(W_{n+1})$, $\sum[d(u) + d(v)]$ and $\sum[d(u).d(v)]$ is

$$n^3 + 29n^2 - 66n \tag{44}$$

$$6n(11n - 21) \tag{45}$$

Summing up the equations corresponding to the sum of degrees, we obtain,

$$\sum[d(u) + d(v)] = 4n^3 + 64n^2 - 102n - 18.$$

Since one edge is shared between a pair of vertices

$$\overline{M_1(Q(W_{n+1})} = 2n^3 + 32n^2 - 51n - 9$$

$$= (2n^3 - 9) + (32n^2 - 44n) - 7n$$

$$\overline{M_1(Q(W_{n+1})} = \overline{M_1(R(W_{n+1})} + (2n^3 - 7n - 9)$$

Adding equations corresponding to the product of degrees, obtained,

$$\sum[d(u).d(v)] = 6n^3 + 123n^2 - 201n - 36.$$

Since one edge is shared between a pair of vertices,

$$\overline{M_2(Q(W_{n+1})} = \frac{1}{2}(6n^3 + 123n^2 - 201n - 36).$$

$$= [3n^3 - 18] + \frac{1}{2}(123n^2 - 201n).$$

$$= [(108n^2 - 188n) + \frac{1}{2}(9n^2 - 27n)] + \frac{1}{2}(6n^2 + 14n) + [3n^3 - 18]$$

$$\overline{M_2(Q(W_{n+1})} = 2\overline{M_2(R(W_{n+1}))} + \overline{M_2(W_{n+1})} + [3n(n^2 - n + 7) - 18].$$

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