Mathematica Aeterna, vol. 3, 2013, no. 6, 449 - 458

Generalized triangle algebras

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Abstract

In this paper, we introduce the notions of generalized triangle algebras in a generalized residuated lattices and give their examples.

Mathematics Subject Classification: 03E72, 03G10, 06A15, 06F07

Keywords:
Generalized residuated lattices, Generalized triangle algebras

1 Introduction

Hájek [8] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Moreover, Georgescu and Popescue [6,7] introduced a generalized residuated lattice which is induced by two implications. By using these concepts, information systems and decision rules are investigated [2,9,12]. Deschrijver, et.al. [3-5,10,11] introduced triangle algebras and interval-valued residuated lattices.

In this paper, we introduce the notions of generalized triangle algebras in a generalized residuated lattices. These concepts are generalizations of triangle algebras and interval-valued residuated lattices. Moreover, we give their examples.

Definition 1.1 [6,7] A structure $(L, \vee, \wedge, \odot, \to, \Rightarrow, \bot, \top)$ is called a *generalized residuated lattice* if it satisfies the following conditions:

(GR1) $(L, \vee, \wedge, \top, \bot)$ is a bounded where $\top$ is the universal upper bound and $\bot$ denotes the universal lower bound;

(GR2) $(L, \odot, \top)$ is a monoid;

(GR3) it satisfies a residuation , i.e.

$$a \odot b \leq c \text{ iff } a \leq b \to c \text{ iff } b \leq a \Rightarrow c.$$

We call that a generalized residuated lattice has the law of double negation if $a = (a^*)^0 = (a^0)^*$ where $a^0 = a \to \bot$ and $a^* = a \Rightarrow \bot$.

Remark 1.2 [6,7,12] (1) A generalized residuated lattice is a residuated lattice ($\to=\Rightarrow$) iff $\odot$ is commutative.

(2) A left-continuous t-norm $([0,1], \leq, \odot)$ defined by $a \to b = \bigvee\{c \mid a \odot c \leq b\}$ is a residuated lattice

(3) A pseudo MV-algebra is a generalized residuated lattice with the law of double negation.

Lemma 1.3 [4,5] Let $(L, \wedge, \vee, \odot, \to, \Rightarrow, \bot, \top)$ be a generalized residuated lattice with the law of double negation.

For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$ for $\to \in \{\to, \Rightarrow\}$.

(2) $x \odot y \leq x \wedge y$.

(3) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$ for $\to \in \{\to, \Rightarrow\}$.

(4) $(x \odot y) \to z = x \to (y \to z)$ and $(x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$.

(5) $(x \odot y)^0 = x \to y^0$ and $(x \odot y)^* = y \Rightarrow x^*$.

(6) $x \to (y \Rightarrow z) = y \Rightarrow (x \to z)$ and $x \Rightarrow (y \to z) = y \to (x \Rightarrow z)$.

(7) $x \odot (x \to y) \leq y$ and $(x \Rightarrow y) \odot x \leq y$.

(8) $(x \Rightarrow y) \odot (y \Rightarrow z) \leq x \Rightarrow z$ and $(y \to z) \odot (x \to y) \leq x \to z$.

(9) $x \to y = \top$ iff $x \leq y$ iff $x \Rightarrow y = \top$.

(10) $x \to y = y^0 \Rightarrow x^0$ and $x \Rightarrow y = y^* \to x^*$.

(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(12) $\bigwedge_{i\in\Gamma} x_i^0 = (\bigvee_{i\in\Gamma} x_i)^0$ and $\bigvee_{i\in\Gamma} x_i^0 = (\bigwedge_{i\in\Gamma} x_i)^0$.

2 Generalized triangle algebras

Definition 2.1 A structure $\mathcal{A} = (A, \wedge, \vee, \odot, \Rightarrow, \to, \nu_i, \mu_i, \bot, e_i, \top)$ for $i \in \{1, 2\}$ is called a *generalized triangle algebra* if it satisfies the following conditions:

(R) $(A, \wedge, \vee, \odot, \Rightarrow, \to, \bot, \top)$ is a generalized residuated lattice.

(T1) $\nu_i(x) \leq x$ and $\nu_i(x) \leq \nu_i(\nu_i(x))$.

(T2) $\nu_i(x \wedge y) = \nu_i(x) \wedge \nu_i(y)$ and $\nu_i(x \vee y) = \nu_i(x) \vee \nu_i(y)$.

(T3) $\nu_i(e_i) = \bot$.

(T4) $\nu_i \circ \mu_i = \mu_i$.

(S1) $x \leq \mu_i(x)$ and $\mu_i(x) \geq \mu_i(\mu_i(x))$.

(S2) $\mu_i(x \wedge y) = \mu_i(x) \wedge \mu_i(y)$ and $\mu_i(x \vee y) = \mu_i(x) \vee \mu_i(y)$.

(S3) $\mu_i(e_i) = \top$.

(S4) $\mu_i \circ \nu_i = \nu_i$.

(T5) $\nu_1(x \Rightarrow y) \leq \nu_1(x) \Rightarrow \nu_1(x)$ and $\nu_2(x \to y) \leq \nu_2(x) \to \nu_1(x)$.

(T6) $(\nu_1(x) \leftrightarrow \nu_1(y)) \odot (\mu_1(x) \leftrightarrow \mu_1(y)) \leq (x \leftrightarrow y)$ and $(\mu_2(x) \Leftrightarrow \mu_2(y)) \odot (\nu_2(x) \Leftrightarrow \nu_2(y)) \leq (x \Leftrightarrow y)$.

(T7) $\nu_1(x) \Rightarrow \nu_1(y) \leq \nu_1(\nu_1(x) \Rightarrow \nu_1(y))$ and $\nu_2(x) \to \nu_2(y) \leq \nu_2(\nu_2(x) \to \nu_2(y))$.

Remark 2.2 (1) If $\odot$ is commutative (or $\Rightarrow=\to$), $\nu_1 = \nu_2$, $\mu_1 = \mu_2$ and $e_1 = e_2$ in Definition 1.4, then $(A, \wedge, \vee, \odot, \Rightarrow, \nu_1, \mu_1, \perp, e_1, \top)$ is a triangle algebra in [10].

(2) In Definition 2.1, $\nu_i(\top) = \top$ and $\mu_i(\perp) = \perp$ because $\nu_i(\top) = \nu_i(\mu_i(e_i)) = \mu_i(e_i) = \top$ and $\mu_i(\perp) = \mu_i(\nu_i(e_i)) = \nu_i(e_i) = \perp$.

Theorem 2.3 *Let $(A, \wedge, \vee, \odot, \Rightarrow, \to, \perp, \top)$ be a generalized residuated lattice with $x^{0*} = x^{*0} = x$ for each $x \in A$. If there exists $e \in A$ such that $e = e^0 = e^*$ and ν_i is an operator satisfying the conditions (T1)-(T3), (T5), (T7),*

$$(\nu_1(x) \leftrightarrow \nu_1(y)) \odot (\nu_1(x^0) \Leftrightarrow \nu_1(y^0)) \leq (x \leftrightarrow y),$$

$$(\nu_2(x^*) \leftrightarrow \nu_2(y^*)) \odot (\nu_2(x) \Leftrightarrow \nu_2(y)) \leq (x \Leftrightarrow y),$$

$$(\nu_i((\nu_i(x))^0))^* = (\nu_i((\nu_i(x))^*))^0$$

and we define

$$\mu_1(x) = (\nu_1(x^0))^*, \ \mu_2(x) = (\nu_2(x^*))^0,$$

then, for $i \in \{1, 2\}$, $(A, \wedge, \vee, \odot, \Rightarrow, \to, \nu_i, \mu_i, \perp, e_i = e, \top)$ is a generalized triangle algebra.

Proof (S1) Since $\nu_1(x^0) \leq x^0$ and $\nu_2(x^*) \leq x^*$, we have

$$x = x^{0*} \leq (\nu_1(x^0))^* = \mu_1(x),$$

$$x = x^{*0} \leq (\nu_2(x^*))^0 = \mu_2(x).$$

Since $\nu_1(x^0) = (\nu_1(x^0))^{*0}$ and $\nu_2(x^*) = (\nu_2(x^*))^{0*}$, we have $\nu_1(x^0) \geq \nu_1(\nu_1(x^0)) = \nu_1((\nu_1(x^0))^{*0})$ and $\nu_2(x^*) \geq \nu_2(\nu_2(x^*)) = \nu_2((\nu_2(x^*))^{0*})$. Thus

$$\mu_1(\mu_1(x)) = (\nu_1((\nu_1(x^0))^{*0}))^* \leq (\nu_1(x^0))^* = \mu_1(x),$$

$$\mu_2(\mu_2(x)) = (\nu_2((\nu_2(x^*))^{0*}))^0 \leq (\nu_2(x^*))^0 = \mu_2(x).$$

(S2) Since $x^{0*} = x^{*0} = x$ for each $x \in X$, we have $\mu_1(x \wedge y) = (\nu_1((x \wedge y)^0))^* = (\nu_1(x^0 \vee y^0))^* = (\nu_1(x^0))^* \wedge (\nu_1(y^0))^* = \mu_1(x) \wedge \mu_1(y)$. Other cases are similarly proved.

(S3)

$$\mu_1(e) = (\nu_1(e^0))^* = (\nu_1(e))^* = \top,$$
$$\mu_2(e) = (\nu_2(e^*))^0 = (\nu_2(e))^0 = \top.$$

(T4)

$$\begin{aligned}\mu_1(x) &= \nu_1(x^0) \Rightarrow \bot = \nu_1(x^0) \Rightarrow \nu_1(e)\\ &= \nu_1(x^0) \Rightarrow \nu_1(\bot) \leq \nu_1(\nu_1(x^0) \Rightarrow \nu_1(\bot)) \quad \text{(by (T7))}\\ &= \nu_1(\nu_1(x^0) \Rightarrow \bot) = \nu_1(\mu_1(x)).\end{aligned}$$

$$\begin{aligned}\mu_2(x) &= \nu_2(x^*) \to \bot = \nu_2(x^*) \to \nu_2(e)\\ &= \nu_2(x^*) \to \nu_2(\bot) \leq \nu_2(\nu_2(x^*) \to \nu_2(\bot)) \quad \text{(by (T7))}\\ &= \nu_2(\nu_2(x^*) \to \bot) = \nu_2(\mu_2(x)).\end{aligned}$$

(S4) Since $\mu_1(x^*) = (\nu_1(x))^*$ and $\nu_1(\mu_1(x^*)) = \nu_1((\nu_1(x))^*) = \mu_1(x^*)$,

$$\begin{aligned}\mu_1(\nu_1(x)) &= (\nu_1(\nu_1(x))^0)^* = (\nu_1(\nu_1(x))^*)^0\\ &= (\mu_1(x^*))^0 = \nu_1(x).\end{aligned}$$

Since $\mu_2(x^0) = (\nu_2(x))^0$ and $\nu_2(\mu_2(x^0)) = \nu_2((\nu_2(x))^0) = \mu_2(x^0)$,

$$\begin{aligned}\mu_2(\nu_2(x)) &= (\nu_2(\nu_2(x))^*)^0 = (\nu_2(\nu_2(x))^0)^*\\ &= (\mu_2(x^0))^* = \nu_2(x).\end{aligned}$$

(T6)

$$\begin{aligned}&(\nu_1(x) \leftrightarrow \nu_1(y)) \odot (\mu_1(x) \leftrightarrow \mu_1(y))\\ &= (\nu_1(x) \leftrightarrow \nu_1(y)) \odot ((\nu_1(x^0))^* \leftrightarrow (\nu_1(y^0))^*)\\ &= (\nu_1(x) \leftrightarrow \nu_1(y)) \odot (\nu_1(x^0) \Leftrightarrow \nu_1(y^0)) \leq (x \leftrightarrow y),\\ &(\mu_2(x) \Leftrightarrow \mu_2(y)) \odot (\nu_2(x) \Leftrightarrow \nu_2(y))\\ &= ((\nu_2(x^*))^0 \Leftrightarrow (\nu_2(y^*))^0) \odot (\nu_2(x) \Leftrightarrow \nu_2(y))\\ &= (\nu_2(x^*) \leftrightarrow \nu_2(y^*)) \odot (\nu_2(x) \Leftrightarrow \nu_2(y)) \leq (x \Leftrightarrow y).\end{aligned}$$

Hence, for $i \in \{1,2\}$, $(A, \wedge, \vee, \odot, \Rightarrow, \to, \nu_i, \mu_i, \bot, e_i = e, \top)$ is a generalized triangle algebra.

Theorem 2.4 *Let $(A, \wedge, \vee, \odot, \Rightarrow, \to, \bot, \top)$ be a generalized residuated lattice with $x^{0*} = x^{*0} = x$ for each $x \in A$. If there exists $e \in A$ such that $e = e^0 = e^*$ and μ_i is an operator satisfying the conditions (S1)-(S3),*

$$\mu_1(\mu_1(y) \odot (\mu_1(x^0))^*) \leq \mu_1(y) \odot (\mu_1(x^0))^* \leq \mu_1(y \odot x)$$

$$\mu_2((\mu_2(x^*))^0 \odot \mu_2(y)) \leq (\mu_2(x^*))^0 \odot \mu_2(y) \leq \mu_2(x \odot y),$$

$$((\mu_1(x^0)) \Leftrightarrow \mu_1(y^0)) \odot (\mu_1(x) \leftrightarrow \mu_1(y)) \leq (x \leftrightarrow y),$$

$$(\mu_2(x) \Leftrightarrow \mu_2(y)) \odot (\mu_2(x^*) \leftrightarrow \mu_2(y^*)) \leq (x \Leftrightarrow y),$$

$$(\mu_i((\mu_i(x))^*))^0 = (\mu_i((\mu_i(x))^0))^*,$$

and we define

$$\nu_1(x) = (\mu_1(x^0))^*,\ \nu_2(x) = (\mu_2(x^*))^0,$$

then, for $i \in \{1, 2\}$, $(A, \wedge, \vee, \odot, \Rightarrow, \rightarrow, \nu_i, \mu_i, \bot, e_i = e, \top)$ *is a generalized triangle algebra.*

Proof (T1) Since $\mu_1(x^0) \geq x^0$ and $\mu_2(x^*) \geq x^*$, we have

$$x = x^{0*} \geq (\mu_1(x^0))^* = \nu_1(x),$$

$$x = x^{*0} \geq (\mu_2(x^*))^0 = \nu_2(x).$$

By (S1), we have

$$\nu_1(\nu_1(x)) = (\mu_1((\mu_1(x^*))^{0*}))^0 \leq (\mu_1(x^*))^0 = \nu_1(x),$$

$$\nu_2(\nu_2(x)) = (\mu_2((\mu_2(x^0))^{*0}))^* \leq (\mu_2(x^0))^* = \nu_2(x).$$

(T2) Since $x^{0*} = x^{*0} = x$ for each $x \in X$, we have $\nu_1(x \wedge y) = (\mu_1((x \wedge y)^*))^0 = (\mu_1(x^* \vee y^*))^0 = (\mu_1(x^*))^0 \wedge (\mu_1(y^*))^0 = \nu_1(x) \wedge \nu_1(y)$. Other cases are similarly proved.

(T3)

$$\nu_1(e) = (\mu_1(e^*))^0 = (\mu_1(e))^0 = \bot,$$

$$\nu_2(e) = (\mu_2(e^0))^* = (\mu_2(e))^* = \bot.$$

(S4) Since $\mu_1(\mu_1(e) \odot (\mu_1(x^0))^*) \leq \mu_1(e) \odot (\mu_1(x^0))^*$, we have

$$\nu_1(x) = (\mu_1(x^0))^* \geq \mu_1((\mu_1(x^0))^* = \mu_1(\nu_1(x))$$

Since $\mu_2((\mu_2(x^*))^0 \odot \mu_2(e)) \leq (\mu_2(x^*))^0 \odot \mu_2(e)$, we have

$$\nu_2(x) = (\mu_2(x^*))^0 \geq \mu_2((\mu_2(x^*))^0 = \mu_2(\nu_2(x))$$

(T4) Since $\nu_1(x^*) = (\mu_1(x))^*$ and $\mu_1(\nu_1(x^*)) = \mu_1((\mu_1(x))^*) = \nu_1(x^*)$,

$$\begin{aligned}\nu_1(\mu_1(x)) &= (\mu_1(\mu_1(x))^0)^* = (\mu_1(\mu_1(x))^*)^0 \\ &= (\nu_1(x^*))^0 = \mu_1(x).\end{aligned}$$

Since $\nu_2(x^0) = (\mu_2(x))^0$ and $\mu_2(\nu_2(x^0)) = \mu_2((\mu_2(x))^0) = \nu_2(x^0)$,

$$\begin{aligned}\nu_2(\mu_2(x)) &= (\mu_2(\mu_2(x))^*)^0 = (\mu_2(\mu_2(x))^0)^* \\ &= (\nu_2(x^0))^* = \mu_2(x).\end{aligned}$$

(T5) Since $y^0 \odot x = (x \Rightarrow y)^0$, we have

$$\begin{array}{l}\mu_1(y^0) \odot (\mu_1(x^0))^* \leq \mu_1(y^0 \odot x)\\ \Leftrightarrow (\mu_1(x^0))^* \Rightarrow (\mu_1(y^0))^* \geq (\mu_1(y^0 \odot x))^*\\ \Leftrightarrow (\mu_1(x^0))^* \Rightarrow (\mu_1(y^0))^* \geq (\mu_1((x \Rightarrow y)^0)^*\\ \Leftrightarrow \nu_1(x) \Rightarrow \nu_1(y) \geq \nu_1(x \Rightarrow y).\end{array}$$

Since $x \odot y^* = (x \rightarrow y)^*$, we have

$$\begin{array}{l}(\mu_2(x^*))^0 \odot \mu_2(y^*) \leq \mu_2(x \odot y^*)\\ \Leftrightarrow (\mu_2(x^*))^0 \rightarrow (\mu_2(y^*))^0 \geq (\mu_2(x \odot y^*))^0\\ \Leftrightarrow (\mu_2(x^*))^0 \rightarrow (\mu_2(y^*))^0 \geq (\mu_2(x \odot y^*))^0\\ \Leftrightarrow \nu_2(x) \rightarrow \nu_2(y) \geq (\mu_2((x \odot y^*)^{0*}))^0 = (\mu_2((x \rightarrow y)^*))^0\\ \Leftrightarrow \nu_2(x) \rightarrow \nu_2(y) \geq \nu_2(x \rightarrow y).\end{array}$$

(T6)

$$\begin{array}{l}(\nu_1(x) \leftrightarrow \nu_1(y)) \odot (\mu_1(x) \leftrightarrow \mu_1(y))\\ = ((\mu_1(x^0))^* \leftrightarrow (\mu_1(y^0))^*) \odot (\mu_1(x) \leftrightarrow \mu_1(y))\\ = ((\mu_1(x^0)) \Leftrightarrow (\mu_1(y^0))) \odot (\mu_1(x) \leftrightarrow \mu_1(y)) \leq (x \leftrightarrow y),\\ (\mu_2(x) \Leftrightarrow \mu_2(y)) \odot (\nu_2(x) \Leftrightarrow \nu_2(y))\\ = (\mu_2(x) \Leftrightarrow \mu_2(y)) \odot ((\mu_2(x^*))^0 \Leftrightarrow (\mu_2(y^*))^0)\\ = (\mu_2(x) \Leftrightarrow \mu_2(y)) \odot (\mu_2(x^*) \leftrightarrow \mu_2(y^*)) \leq (x \Leftrightarrow y).\end{array}$$

(T7) Since $b \odot a = (a \Rightarrow b^*)^0$, we have

$$\begin{array}{l}\mu_1(\mu_1(y^0) \odot (\mu_1(x^0))^*) \leq \mu_1(y^0) \odot (\mu_1(x^0))^*\\ \Leftrightarrow \mu_1(((\mu_1(x^0))^* \Rightarrow \mu_1(y^0)^*)^0) \leq \mu_1(y^0) \odot (\mu_1(x^0))^*\\ \Leftrightarrow \mu_1(((\nu_1(x) \Rightarrow \nu_1(y))^0) \leq \mu_1(y^0) \odot (\mu_1(x^0))^*\\ \Leftrightarrow (\mu_1(x^0))^* \Rightarrow (\mu_1(y^0))^* \leq \mu_1(((\nu_1(x) \Rightarrow \nu_1(y))^0)^*\\ \Leftrightarrow \nu_1(x) \Rightarrow \nu_2(y) \leq \nu_1(\nu_1(x) \Rightarrow \nu_1(y)),\end{array}$$

Since $(\mu_2(x^*))^0 \odot \mu_2(y^*) = ((\mu_2(x^*))^0 \rightarrow \mu_2(y^*)^0)^*$, we have

$$\begin{array}{l}\mu_2((\mu_2(x^*))^0 \odot \mu_2(y^*)) \leq (\mu_2(x^*))^0 \odot \mu_2(y^*)\\ \Leftrightarrow \mu_2(((\mu_2(x^*))^0 \rightarrow \mu_2(y^*)^0)^*) \leq (\mu_2(x^*))^0 \odot \mu_2(y^*)\\ \Leftrightarrow \mu_2(((\nu_2(x) \rightarrow \nu_2(y))^*) \leq (\mu_2(x^*))^0 \odot \mu_2(y^*)\\ \Leftrightarrow (\mu_2(x^*))^0 \rightarrow (\mu_2(y^*))^0 \leq \mu_2(((\nu_2(x) \rightarrow \nu_2(y))^*)^0\\ \Leftrightarrow \nu_2(x) \rightarrow \nu_2(y) \leq \nu_2(\nu_2(x) \rightarrow \nu_2(y)).\end{array}$$

Example 2.5 Let $K = \{(x, y) \in R^2 \mid x > 0\}$ be a set and we define an operation $\otimes : K \times K \rightarrow K$ as follows:

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1x_2, x_1y_2 + y_1).$$

Then $(K, \otimes)$ is a group with $e = (1,0)$, $(x,y)^{-1} = (\frac{1}{x}, -\frac{y}{x})$.

We have a positive cone $P = \{(a,b) \in R^2 \mid a = 1, b \geq 0 \text{ , or } a > 1\}$ because $P \cap P^{-1} = \{(1,0)\}$, $P \odot P \subset P$, $(a,b)^{-1} \odot P \odot (a,b) = P$ and $P \cup P^{-1} = K$. For $(x_1,y_1),(x_2,y_2) \in K$, we define

$$\begin{aligned}(x_1,y_1) \leq (x_2,y_2) \quad &\Leftrightarrow (x_1,y_1)^{-1} \odot (x_2,y_2) \in P,\ (x_2,y_2) \odot (x_1,y_1)^{-1} \in P \\ &\Leftrightarrow x_1 < x_2 \ \text{ or } \ x_1 = x_2, y_1 \leq y_2.\end{aligned}$$

Then $(K, \leq \otimes)$ is a lattice-group.

The structure $(L, \odot, \Rightarrow, \rightarrow, (\frac{1}{2}, 1), (1,0))$ is a generalized residuated lattice with strong negation where $\bot = (\frac{1}{2}, 1)$ is the least element and $\top = (1,0)$ is the greatest element from the following statements:

$$\begin{aligned}(x_1,y_1) \odot (x_2,y_2) \quad &= (x_1,y_1) \otimes (x_2,y_2) \vee (\tfrac{1}{2}, 1) = (x_1x_2, x_1y_2 + y_1) \vee (\tfrac{1}{2}, 1), \\ (x_1,y_1) \Rightarrow (x_2,y_2) \quad &= ((x_1,y_1)^{-1} \otimes (x_2,y_2)) \wedge (1,0) = (\tfrac{x_2}{x_1}, \tfrac{y_2-y_1}{x_1}) \wedge (1,0), \\ (x_1,y_1) \rightarrow (x_2,y_2) \quad &= ((x_2,y_2) \otimes (x_1,y_1)^{-1}) \wedge (1,0) = (\tfrac{x_2}{x_1}, -\tfrac{x_2y_1}{x_1} + y_2) \wedge (1,0).\end{aligned}$$

Furthermore, we have $(x,y) = (x,y)^{*\circ} = (x,y)^{\circ *}$ from:

$$(x,y)^* = (x,y) \Rightarrow (\frac{1}{2}, 1) = (\frac{1}{2x}, \frac{1-y}{x}),$$

$$(x,y)^{*\circ} = (\frac{1}{2x}, \frac{1-y}{x}) \rightarrow (\frac{1}{2}, 1) = (x,y).$$

(1) There exists $e = (\frac{1}{\sqrt{2}}, 2 - \sqrt{2}) = e^0 = e^*$. For $i \in \{1,2\}$, let ν_i be an operator satisfying the conditions in Theorem 2.3. Define

$$\mu_1(x) = (\nu_1(x^0))^*,\ \mu_2(x) = (\nu_2(x^*))^0.$$

For $i \in \{1,2\}$, $(L, \wedge, \vee, \odot, \Rightarrow, \rightarrow, \nu_i, \mu_i, \bot, e_i = e, \top)$ is a generalized triangle structure.

(2) There exists $e = (\frac{1}{\sqrt{2}}, 2 - \sqrt{2}) = e^0 = e^*$. For $i \in \{1,2\}$, let μ_i be an operator satisfying the conditions in Theorem 2.4. Define

$$\nu_1(x) = (\mu_1(x^0))^*,\ \nu_2(x) = (\mu_2(x^*))^0.$$

For $i \in \{1,2\}$, $(L, \wedge, \vee, \odot, \Rightarrow, \rightarrow, \nu_i, \mu_i, \bot, e_i = e, \top)$ is a generalized triangle structure.

(3) We define

$$L^{[2]} = \{[(x_1,y_1),(x_2,y_2)] \mid ((x_1,y_1),(x_2,y_2)) \in L \times L,\ (x_1,y_1) \leq (x_2,y_2)\}$$

$$e = [(\frac{1}{2}, 1), (1,0)] = [\bot, \top],$$

$$[(x_1,y_1),(x_2,y_2)] \wedge [(x_3,y_3),(x_4,y_4)] = [(x_1,y_1) \wedge (x_3,y_3), (x_2,y_2) \wedge (x_4,y_4)]$$

$$[(x_1,y_1),(x_2,y_2)]\vee[(x_3,y_3),(x_4,y_4)]=[(x_1,y_1)\vee(x_3,y_3),(x_2,y_2)\vee(x_4,y_4)]$$

$$[(x_1,y_1),(x_2,y_2)]\odot[(x_3,y_3),(x_4,y_4)]=[(x_1,y_1)\odot(x_3,y_3),(x_2,y_2)\odot(x_4,y_4)]$$

$$\begin{array}{l}[(x_1,y_1),(x_2,y_2)]\Rightarrow[(x_3,y_3),(x_4,y_4)]\\ =[\Big(((x_1,y_1)\Rightarrow(x_3,y_3))\wedge((x_2,y_2)\Rightarrow(x_4,y_4)),(x_2,y_2)\Rightarrow(x_4,y_4)]\\ [(x_1,y_1),(x_2,y_2)]\rightarrow[(x_3,y_3),(x_4,y_4)]\\ =[\Big(((x_1,y_1)\rightarrow(x_3,y_3))\wedge((x_2,y_2)\rightarrow(x_4,y_4)),(x_2,y_2)\rightarrow(x_4,y_4)].\end{array}$$

Moreover, we define $\nu_1=\nu_2,\mu_1=\mu_2:L^{[2]}\to L^{[2]}$ as

$$\nu_1([(x_1,y_1),(x_2,y_2)])=[(x_1,y_1),(x_1,y_1)],$$

$$\mu_1([(x_1,y_1),(x_2,y_2)])=[(x_2,y_2),(x_2,y_2)].$$

Then $(L^{[2]},\wedge,\vee,\odot,\Rightarrow,\rightarrow,\nu_i,\mu_i,[\bot,\bot],e,[\top,\top])$ for $i\in\{1,2\}$ is a generalized triangle algebra from the following statements.

(R) $(L^{[2]},\wedge,\vee,\odot,\Rightarrow,\rightarrow,[\bot,\bot],[\top,\top])$ is a generalized residuated lattice.

$$\begin{array}{l}[(x_1,y_1),(x_2,y_2)]\odot[(x_3,y_3),(x_4,y_4)]\le[(x_5,y_5),(x_6,y_6)]\\ \text{iff }\ [(x_1,y_1)\odot(x_3,y_3),(x_2,y_2)\odot(x_4,y_4)]\le[(x_5,y_5),(x_6,y_6)]\\ \text{iff }\ (x_1,y_1)\odot(x_3,y_3)\le(x_5,y_5),\ \ (x_2,y_2)\odot(x_4,y_4)\le(x_6,y_6)\\ \text{iff }\ (x_1,y_1)\le(x_3,y_3)\rightarrow(x_5,y_5),\ \ (x_2,y_2)\le(x_4,y_4)\rightarrow(x_6,y_6)\\ \text{iff }\ [(x_1,y_1),(x_2,y_2)]\le[((x_3,y_3)\rightarrow(x_5,y_5)\wedge(x_4,y_4)\rightarrow(x_6,y_6))\\ \quad\ ,(x_4,y_4)\rightarrow(x_6,y_6)]\\ \text{iff }\ [(x_1,y_1),(x_2,y_2)]\le[(x_3,y_3),(x_4,y_4)]\rightarrow[(x_5,y_5),(x_6,y_6)].\end{array}$$

(T1) $\nu_i([(x_1,y_1),(x_2,y_2)])=[(x_1,y_1),(x_1,y_1)]\le[(x_1,y_1),(x_2,y_2)]$ and

$$\nu_i([(x_1,y_1),(x_2,y_2)])=\nu_i(\nu_i([(x_1,y_1),(x_2,y_2)]))=[(x_1,y_1),(x_1,y_1)].$$

(T2) It is easily proved.

(T3) $\nu_i(e)=\nu_i([\bot,\top])=\bot$.

(T4)

$$\mu_i([(x_1,y_1),(x_2,y_2)])=[(x_2,y_2),(x_2,y_2)])=\nu_i(\mu_i([(x_1,y_1),(x_2,y_2)]).$$

(T5) $\nu_1([(x_1,y_1),(x_2,y_2)]\Rightarrow[(x_3,y_3),(x_4,y_4)])\le\nu_1([(x_1,y_1),(x_2,y_2)])\Rightarrow\nu_1([(x_3,y_3),(x_4,y_4)])$ from:

$$\begin{array}{l}\nu_1([(x_1,y_1),(x_2,y_2)]\Rightarrow[(x_3,y_3),(x_4,y_4)])\\ =\nu_1([\Big(((x_1,y_1)\Rightarrow(x_3,y_3))\wedge((x_2,y_2)\Rightarrow(x_4,y_4))\Big),(x_2,y_2)\Rightarrow(x_4,y_4)])\\ =[\Big(((x_1,y_1)\Rightarrow(x_3,y_3))\wedge((x_2,y_2)\Rightarrow(x_4,y_4))\Big),\\ \Big(((x_1,y_1)\Rightarrow(x_3,y_3))\wedge((x_2,y_2)\Rightarrow(x_4,y_4))\Big)].\end{array}$$

$$\begin{aligned}&\nu_1([(x_1,y_1),(x_2,y_2)]) \Rightarrow \nu_1([(x_3,y_3),(x_4,y_4)])\\&= [(x_1,y_1),(x_1,y_1)] \Rightarrow [(x_3,y_3),(x_3,y_3)]\\&= [(x_1,y_1) \Rightarrow (x_3,y_3),((x_1,y_1) \Rightarrow (x_3,y_3))].\end{aligned}$$

Similarly, $\nu_1([(x_1,y_1),(x_2,y_2)] \to [(x_3,y_3),(x_4,y_4)]) \le \nu_1([(x_1,y_1),(x_2,y_2)]) \to \nu_1([(x_3,y_3),(x_4,y_4)])$.

(T6) $(\nu_1([(x_1,y_1),(x_2,y_2)]) \leftrightarrow \nu_1([(x_3,y_3),(x_4,y_4)]))\odot(\mu_1([(x_1,y_1),(x_2,y_2)]) \leftrightarrow \mu_1([(x_3,y_3),(x_4,y_4)])) \le [(x_1,y_1),(x_2,y_2)] \leftrightarrow [(x_3,y_3),(x_4,y_4)]$ from:

$$\begin{aligned}&(\nu_1([(x_1,y_1),(x_2,y_2)]) \leftrightarrow \nu_1([(x_3,y_3),(x_4,y_4)]))\\&\odot(\mu_1([(x_1,y_1),(x_2,y_2)]) \leftrightarrow \mu_1([(x_3,y_3),(x_4,y_4)]))\\&= ([(x_1,y_1),(x_1,y_1)] \leftrightarrow [(x_3,y_3),(x_3,y_3)])\\&\odot([(x_2,y_2),(x_2,y_2)] \leftrightarrow [(x_4,y_4),(x_4,y_4)])\\&= ([(x_1,y_1) \leftrightarrow (x_3,y_3),(x_1,y_1) \leftrightarrow (x_3,y_3)])\\&\odot([(x_2,y_2) \leftrightarrow (x_4,y_4),(x_2,y_2) \leftrightarrow (x_4,y_4)])\\&= [((x_1,y_1) \leftrightarrow (x_3,y_3)) \odot ((x_2,y_2) \leftrightarrow (x_4,y_4)),\\&((x_1,y_1)) \leftrightarrow (x_3,y_3)) \odot ((x_2,y_2) \leftrightarrow (x_4,y_4))]\\&\le [((x_1,y_1) \leftrightarrow (x_3,y_3)) \wedge ((x_2,y_2) \leftrightarrow (x_4,y_4)),(x_2,y_2) \leftrightarrow (x_4,y_4)]\\&= [(x_1,y_1),(x_2,y_2)] \leftrightarrow [(x_3,y_3),(x_4,y_4)].\end{aligned}$$

Similarly, $(\nu_2([(x_1,y_1),(x_2,y_2)]) \Leftrightarrow \nu_2([(x_3,y_3),(x_4,y_4)]))\odot(\mu_2([(x_1,y_1),(x_2,y_2)]) \Leftrightarrow \mu_2([(x_3,y_3),(x_4,y_4)])) \le [(x_1,y_1),(x_2,y_2)] \Leftrightarrow [(x_3,y_3),(x_4,y_4)]$.

(T7)

$$\begin{aligned}&\nu_1([(x_1,y_1),(x_2,y_2)]) \Rightarrow \nu_1([(x_3,y_3),(x_4,y_4)])\\&= [(x_1,y_1) \Rightarrow (x_3,y_3),((x_1,y_1) \Rightarrow (x_3,y_3))]\\&= \nu_1(\nu_1([(x_1,y_1),(x_2,y_2)]) \Rightarrow \nu_1([(x_3,y_3),(x_4,y_4)])).\end{aligned}$$

Other case is similarly proved.

(S4)

$$\nu_i([(x_1,y_1),(x_2,y_2)]) = [(x_1,y_1),(x_1,y_1)]) = \mu_i(\nu_i([(x_1,y_1),(x_2,y_2)]).$$

(S1), (S2) and (S3) are similarly proved as (T1),(T2) and (T3), respectively.

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Received: June, 2013