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TWO VERY SPECIAL PYTHAGOREAN TRIANGLES

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Abstract: *There are various Special Pythagorean Triangles with their areas as Triangular numbers. There are also Pythagorean Triangles with their areas as Pentagonal numbers. This paper investigates the existence of Special Pythagorean Triangles with their areas as both Triangular and Pentagonal Numbers.*

Key words: Pythagorean Triangles, Triangular Numbers, Pentagonal Numbers.

Subject Classification Code: 11D25, 11-04 and 11Y50

1. Introduction

Who has not heard of Pythagorean Theorem? An apparently simple theorem, which fascinated Fermat in the seventeenth century, still continues to motivate the minds of enthusiastic mathematicians throughout the world. The problems related to it keep on engrossing all those who love to have fun with numbers. In Pythagorean theory of numbers, Triangular numbers and Pentagonal numbers played very important role. Rana and Darbari [1] obtained special Pythagorean Triangles, with their legs to be consecutive, in terms of Triangular Numbers and Darbari [2] found special Pythagorean Triangles, with their legs to be consecutive, with their perimeters as Pentagonal Numbers. Special Pythagorean Triangles are generated by Gopalan and Janaki [3]. An attempt has been made to find out special Pythagorean triangles with their area as Triangular Number as well as Pentagonal numbers.

2. Method of Analysis:

The primitive solutions of the Pythagorean Equation,

$$X^2 + Y^2 = Z^2 \qquad (2.1)$$

is given by [4] $$X = m^2 - n^2, Y = 2mn, Z = m^2 + n^2 \qquad (2.2)$$

for some integers ***m*, *n*** of opposite parity such that $m > n > 0$ and $(m, n) = 1$.

2.1 Area is a Triangular as well as Pentagonal number:

Definition 2.1: A natural number is called a Triangular number if it can be written in the form $\frac{\gamma(\gamma+1)}{2}, \gamma \in \mathrm{N}$

Definition 2.2: A natural number is called a pentagonal number if it can be written in the form β (3β- 1)/2, β ε □.

There are just 18 numbers which are both Pentagonal and Triangular less than 10^{20}. They are as follows:

S.N.	β	γ	$\frac{\gamma(\gamma+1)}{2} = \frac{\beta(3\beta-1)}{2}$
1.	1	1	1
2.	12	20	210
3.	165	285	40755
4.	2296	3976	7906276
5.	31977	55385	1533776805
6.	445380	771420	297544793910
7.	6203341	10744501	57722156241751
8.	86401392	149651600	11197800766105800
9.	1203416145	2084377905	2172315626468283465
10.	16761424636	29031639076	421418033734080886426
11.	233456528757	404358569165	81752926228785223683195
12.	3251629977960	5631988329240	15859646270350599313653420
13.	45289363162681	78443478040201	3076689623521787481625080301
14.	630799454299572	1092576704233580	596861927316956420835951924990
15.	8785902997031325	15217630381229925	115788137209866023854693048367775
16.	122371842504138976	211954248632985376	22462301756786691671389615431423376
17.	1704419892060914337	2952141850480565345	4357570752679408318225730700647767185
18.	23739506646348661740	41118031658094929460	845346263718048427044120366310235410530

Table 2.1: Numbers which are both Triangular and Pentagonal

If the area of the Pythagorean Triangle (X, Y, Z) is Triangular number *A*, then

$$\frac{1}{2}XY = \frac{\gamma(\gamma+1)}{2} = A \quad (2.3)$$

And if the area of the Pythagorean Triangle (X, Y, Z) is Pentagonal number *A*, then

$$\tfrac{1}{2} X Y = \beta(3\beta - 1)/2 = A. \quad (2.4)$$

If area is both Triangular and pentagonal number, then by virtue of equations (2.2) and equation (2.3), equation (2.4) becomes

$$(m^2 - n^2)mn = \frac{\gamma(\gamma+1)}{2} = \frac{\beta(3\beta-1)}{2}, \beta, \gamma \in \mathrm{N} \quad (2.4)$$

$$\Rightarrow \beta = \frac{(-1+\sqrt{1+8m^3n-8mn^3})}{2} \quad \text{and} \quad \gamma = \frac{(-1+\sqrt{1+8m^3n-8mn^3})}{2} \quad (2.5)$$

Solving equation (2.5) using *Mathematica* for $0 < m < 10^5$, $0 < n < 10^5$, $0 < \beta < 10^6$, only two special Pythagorean Triangles are obtained with their areas as Triangular Numbers and Pentagonal numbers both. The following tables give these Primitive Pythagorean Triangles:

S.N.	**m**	n	β	γ	**X**	**Y**	**Z**	**X^2**	**Y^2**	**$X^2 + Y^2 = Z^2$**	A= (½)X Y = β (3β - 1)/2 = **γ(γ + 1)/2**
1	**5**	**2**	**12**	20	21	20	29	**441**	**400**	**841**	**210**
2	**6**	**1**	**12**	20	35	12	37	**1225**	**144**	**1369**	**210**

Table 2.2: (X, Y, Z) with XY/2 = β(3β - 1)/2 = γ(γ + 1)/2

4. Observations and conclusion: We observe that

1. For m = 5, n = 2, β = 12, γ = 20, we get very special Pythagorean Triangle with two legs consecutives and its area as Triangular as well as Pentagonal number.
2. X + Y + Z = 0(mod 7).
3. X + Y + Z = 0(mod 14).
4. (X + Y + Z)(X + Y – Z) = 0(mod 2)

5. $(X + Y + Z)(X + Y - Z) = 0(\text{mod } 3)$

6. $(X + Y + Z)(X + Y - Z) = 0(\text{mod } 4)$

7. $(X + Y + Z)(X + Y - Z) = 0(\text{mod } 5)$

8. $(X + Y + Z)(X + Y - Z) = 0(\text{mod } 6)$

9. $(X + Y + Z)(X + Y - Z) = 0(\text{mod} 7)$

10. $(X + Y + Z)(X + Y - Z) = 0(\text{mod } 8)$

11. $(Y + Z - X) = 0(\text{mod } 2)$

12. $(X + 2Y + Z)^2 = (Z - X)^2 + 4(X + Y) (Y + Z)$.

13. $(Y + Z - X)^2 = 2(Y + Z) (Z - X)$.

As conclusion, other special Pythagorean Triangle can be obtained which satisfy the some other conditions discussed in this paper.

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