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Relative Entropy and Mutual Information on a Quantum Logic

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Abstract

The present paper deals with the study of relative entropy, mutual information and their properties in a quantum space (L, s), where L is an orthomodular lattice and s is a Bayesian state on it.

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1 Introduction

Relative entropy plays an important role, as a mathematical device, in the stability analysis of master equations [16] and Fokker-Planck equations [15], and in isothermal equilibrium fluctuations and transient nonequilibrium deviations [14] (see also [4], [15]). To study noncompatible random events in quantum statistical mechanics, Birkhoff and Von Neumann [3] proposed a new mathematical model, called quantum logic. The notion of entropy of partitions in the context of Boolean algebras is a useful tool in studying dynamical systems and has been applied on many other structures ([1], [13], [15]). Yuan [17] introduced the entropy of partitions on quantum logic (or a orthomodular lattice) using the notion of a state (or a probability measure) and Yue-Xu and Zhi-Hao [18] studied conditional entropy of partitions on the same structure.

In the present paper, we employ the theory of entropy and conditional entropy of partitions of a quantum space of orthomodular lattices with Bayes' property, as developed in [6], [7], and put forward the notions of relative entropy and mutual information. The results proved in the present paper generalize the corresponding results of classical quantum space to that on orthomodular lattices. Prerequisites for the paper are collected in Section 2. We introduce and study the notion of relative entropy for a given partition of a quantum space in Section 3. Various useful properties of relative entropy are proved and its relation with mutual information is explored. In particular, convexity of relative entropy with respect to Bayesian states, chain rules for entropy and that for mutual information are established. We have proved the data processing inequality in the framework of quantum logic that may form foundation for the corresponding theory of sufficient statistics.

2 Preliminaries and basic notions

2.1 ([2], [5]). Let $L = \{L, 0, 1, \vee, \wedge\}$ be a bounded poset, where 0 and 1 are smallest and greatest elements in L. An *orthocomplementation* on L is a unary operation $'$ on L satisfying, for $a, b \in L$:

(i) $a \leq b \Longrightarrow b' \leq a'$,

(ii) $(a')' = a$,

(iii) the supremum $a \vee a'$ and the infimum $a \wedge a'$ exist; the equations $a \vee a' = 1$ and $a \wedge a' = 0$ hold.

An *orthoposet* L is a bounded poset with an orthocomplementation. The relation *orthogonal* $\perp$ for elements a, b of an orthoposet L is defined by $a \perp b$ (a is *orthogonal* to b) if $a \leq b'$ holds. An *ortholattice* is an orthoposet which is also a lattice. An *orthomodular lattice* (abbr. *OML*) is an ortholattice satisfying orthomodular law:

$$a, b \in L,\ a \leq b \Longrightarrow b = a \vee (a' \wedge b).$$

Orthomodularity is a weaker form of modularity, which holds for orthogonal elements. The orthomodular law is a kind of distributivity: for $a \leq b$, we have $a \vee (a' \wedge b) = b = 1 \wedge b = (a \vee a') \wedge (a \vee b)$. If an OML L satisfies: $a \wedge b = 0 \Longrightarrow a \leq b'$, then L is a Boolean algebra (see [2], [5]).

2.2. A map $s : L \longrightarrow [0, 1]$ such that

(i) $s(0) = 0$, and

(ii) $a, b \in L$, $a \perp b \Longrightarrow s(a \vee b) = s(a) + s(b)$,

is called a *state* on L. It may be observed that $s(1) = 1, s$ is monotone and $s(a') = 1 - s(a), a \in L$.

We denote by N the set of all natural numbers.

2.3. A (finite) system $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ $(n \in N)$ of elements of an OML L is said to be a *partition* of L corresponding to a state s defined on L (or simply a *partition of the couple* (L, s)) if

(i) $\mathcal{A}$ is a $\vee$-orthogonal system, i.e. $(\bigvee_{i=1}^{k} a_i) \perp a_{k+1}$ for $k = 1, 2, 3, \ldots, n-1$,

(ii) $s(\bigvee_{i=1}^{n} a_i) = 1$.

Let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ $(n, m \in N)$ be partitions of (L, s). Then $\mathcal{A}$ and $\mathcal{B}$ are called *independent* if, $s(a_i \wedge b_j) = s(a_i)s(b_j)$, where $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$. The *common refinement* of partitions $\mathcal{A}$ and $\mathcal{B}$ is defined as $\mathcal{A} \vee \mathcal{B} := \{a_i \wedge b_j : a_i \in \mathcal{A}, b_j \in \mathcal{B}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\}$.

The common refinement $\mathcal{A} \vee \mathcal{B}$ of partitions $\mathcal{A}$ and $\mathcal{B}$ turns out to be a partition of L corresponding to state s, provided s has the *Bayes' property* (or s is *Bayesian*): $s(\bigvee_{j=1}^{m}(a \wedge b_j)) = s(a)$ for every $a \in L$. For, $s(\bigvee_{i=1}^{n} \bigvee_{j=1}^{m}(a_i \wedge b_j)) = \sum_{i=1}^{n} s(\bigvee_{j=1}^{m}(a_i \wedge b_j)) = \sum_{i=1}^{n} s(a_i) = s(\bigvee_{i=1}^{n} a_i) = s(1)$. We call the couple (L, s) a *quantum space* if s has the Bayes' property.

Let us recall the following *log sum inequality*, which we shall use in the sequel to establish various results: *for nonnegative real numbers,* $x_1, x_2, \ldots, x_n$ *and* $y_1, y_2, \ldots, y_n$,

$$\sum_{i=1}^{n} x_i \log \frac{x_i}{y_i} \geq (\sum_{i=1}^{n} x_i) \log \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i};$$

equality holds if and only if $\frac{x_i}{y_i}$ *is constant.* Here we follow the convention that $x \log \frac{x}{0} = \infty$ if $x > 0$, and $0 \log \frac{0}{0} = 0$.

3 Quantum relative entropy and mutual information

Let the systems $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ be partitions of a couple (L, s). Then the *entropy* $H_s(\mathcal{A})$ of the partition $\mathcal{A}$ with respect to s is defined by

$$H_s(\mathcal{A}) := -\sum_{i=1}^{n} g(s(a_i)),$$

where $g : [0, \infty] \longrightarrow R$ is the convex function, called *Shannon's function*, given by $g(x) = x \log x$, if $x > 0$ and $g(0) = 0$. The *conditional entropy* $H_s(\mathcal{A}|\mathcal{B})$ is defined by

$$H_s(\mathcal{A}|\mathcal{B}) := -\sum_{j=1}^{m}\sum_{i=1}^{n} s(b_j)g(s(a_i|b_j)).$$

Notice that $H_s(\mathcal{A}|\mathcal{B}) \geq 0$, and $H_s(\mathcal{A}|\mathcal{A}) = 0$.

We refer to [6], [7] where a study of entropy and conditional entropy of partitions of a couple (L, s), (here s is a state on the OML L) is made, and its relation with the theory of commutators, boolean quotients in orthomodular lattices [2], [8], [9], [10], and Bell inequalities [11], [12], is discussed. Now we recall the following results from [6] that are used in the sequel.

Theorem 3.1 *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be partitions of a quantum space (L, s). Then*

1. $H_s(\mathcal{A} \vee \mathcal{B}) \leq H_s(\mathcal{A}) + H_s(\mathcal{B})$; *equality holds if $\mathcal{A}$ and $\mathcal{B}$ are independent partitions of (L, s).*

2. $H_s(\mathcal{A}|\mathcal{B}) = H_s(\mathcal{A})$ *if and only if $\mathcal{A}$ and $\mathcal{B}$ are independent partitions of (L, s).*

3. $H_s(\mathcal{A}\vee\mathcal{B}) = H_s(\mathcal{A})+H_s(\mathcal{B}|\mathcal{A})$, *and hence* $H_s(\mathcal{A}\vee\mathcal{B}) \geq \max\{H_s(\mathcal{A}), H_s(\mathcal{B})\}$.

4. $H_s(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) = H_s(\mathcal{A}|\mathcal{C}) + H_s(\mathcal{B}|\mathcal{A} \vee \mathcal{C})$.

5. $H_s(\mathcal{A}|\mathcal{B} \vee \mathcal{C}) \leq H(\mathcal{A}|\mathcal{B})$.

Theorem 3.2 (*Concavity of entropy*). *Let L be an OML and r and s be states on it. If $\mathcal{A}$ is a partition of L corresponding to r and s, then for $\alpha \in [0, 1]$, we have*

$$\alpha H_s(\mathcal{A}) + (1 - \alpha)H_r(\mathcal{A}) \leq H_{\alpha s+(1-\alpha)r}(\mathcal{A}),$$

showing that $\mathcal{H}_s(\mathcal{A})$ is a concave function of s.

Proof. Let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ be a partition of (L, r) and (L, s). Then

$$\begin{aligned}
\alpha H_s(\mathcal{A}) + (1 - \alpha)H_r(\mathcal{A}) &= -\alpha\sum_{i=1}^{n} g(s(a_i)) - (1 - \alpha)\sum_{i=1}^{n} g(r(a_i)) \\
&= -\sum_{i=1}^{n}[\alpha g(s(a_i)) + (1 - \alpha)g(r(a_i))] \\
&\leq -\sum_{i=1}^{n} g(\alpha(s(a_i)) + (1 - \alpha)(r(a_i))) \\
&= -\sum_{i=1}^{n} g((\alpha s + (1 - \alpha)r)(a_i)) \\
&= H_{\alpha s+(1-\alpha)r}(\mathcal{A}).
\end{aligned}$$

Theorem 3.3 (*Chain rules for entropy*). *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ $(n \in N)$, and $\mathcal{C}$ be partitions of a quantum space (L, s). Then*

(i) $H_s(\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n) = \sum_{i=1}^{n} H_s(\mathcal{A}_i \mid (\mathcal{A}_{i-1} \vee \cdots \vee \mathcal{A}_1))$.

(ii) $H_s(\bigvee_{i=1}^{n} \mathcal{A}_i|\mathcal{C}) = \sum_{i=1}^{n} H_s(\mathcal{A}_i| \bigvee_{k=1}^{i-1} \mathcal{A}_k \vee \mathcal{C})$.

(iii) $H_s(\mathcal{A}_1 \vee \mathcal{A}_2 \vee \mathcal{A}_3) \leq H_s(\mathcal{A}_1 \vee \mathcal{A}_2) + H_s(\mathcal{A}_1 \vee \mathcal{A}_3) - H_s(\mathcal{A}_1)$.

Proof. (i). By Theorem 3.1(3) and (4), we have $H_s(\mathcal{A}_1 \vee \mathcal{A}_2) = H_s(\mathcal{A}_1) + H_s(\mathcal{A}_2 \mid \mathcal{A}_1)$. For $n = 3$, $H_s(\mathcal{A}_1 \vee \mathcal{A}_2 \vee \mathcal{A}_3) = H_s(\mathcal{A}_1) + H_s(\mathcal{A}_2 \vee \mathcal{A}_3 \mid \mathcal{A}_1) = H_s(\mathcal{A}_1) + H_s(\mathcal{A}_2 \mid \mathcal{A}_1) + H_s(\mathcal{A}_3 \mid \mathcal{A}_1 \vee \mathcal{A}_2)$. Now suppose that the result is true for a specific value of $n \in N$. Then

$$
\begin{aligned}
&H_s(\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n \vee \mathcal{A}_{n+1}) \\
&\quad = H_s(\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n) + H_s(\mathcal{A}_{n+1} \mid (\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n)) \\
&\quad = \sum_{i=1}^{n} H_s(\mathcal{A}_i \mid (\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_{i-1})) + H_s(\mathcal{A}_{n+1} \mid (\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n)) \\
&\quad = \sum_{i=1}^{n+1} H_s(\mathcal{A}_i \mid (\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_{i-1})).
\end{aligned}
$$

Proof of (ii) follows similarly, using Theorem 3.1(4) inductively.

(iii) By Theorem 3.1(3) and (5), we get

$$
\begin{aligned}
H_s(\mathcal{A}_1 \vee \mathcal{A}_2 \vee \mathcal{A}_3) &= H_s(\mathcal{A}_1 \vee \mathcal{A}_2) + H_s(\mathcal{A}_3 \mid \mathcal{A}_1 \vee \mathcal{A}_2) \\
&\leq H_s(\mathcal{A}_3 \mid \mathcal{A}_1) + H_s(\mathcal{A}_1 \vee \mathcal{A}_2) \\
&= H_s(\mathcal{A}_1 \vee \mathcal{A}_2) + H_s(\mathcal{A}_1 \vee \mathcal{A}_3) - H_s(\mathcal{A}_1).
\end{aligned}
$$

Definition 3.4 *Let s_1 and s_2 be states on an OML L, and let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ be a partition of L corresponding to both s_1 and s_2. Then the relative entropy $D_{\mathcal{A}}(s_1 \parallel s_2)$ is defined as*

$$D_{\mathcal{A}}(s_1 \parallel s_2) := \sum_{i=1}^{n} s_1(a_i) \log \frac{s_1(a_i)}{s_2(a_i)}.$$

The following result suggests interpretation of relative entropy as a distance between two states, i.e. a measure of how different the two states are. Due to non-availability of symmetry and the triangle inequality, it is not a metric in a true sense.

Theorem 3.5 *If $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ is a partition of L corresponding to states s_1 and s_2, then $D_{\mathcal{A}}(s_1 \parallel s_2) \geq 0$, with equality if and only if $s_1(a_i) = s_2(a_i)$, for each $i \in \{1, 2, \ldots, n\}$.*

Proof. In the log sum inequality, let $x_i = s_1(a_i)$ and $y_i = s_2(a_i)$ for $i \in \{1, 2, \dots, n\}$. Then $\sum_{i=1}^n x_i = \sum_{i=1}^n s_1(a_i) = s_1(\bigvee_{i=1}^n a_i) = 1$. Similarly, $\sum_{i=1}^n y_i = 1$. Hence $D_{\mathcal{A}}(s_1 \parallel s_2) \geq 0$. Also, $D_{\mathcal{A}}(s_1 \parallel s_2) = 0$ if and only if $s_1(a_i) = \alpha s_2(a_i) \forall i$, where α is a constant. Summing over all i, we get $\alpha = 1$. Thus $D_{\mathcal{A}}(s_1 \parallel s_2) = 0$ if and only if $s_1(a_i) = s_2(a_i), \forall i$.

Theorem 3.6 *Let $\mathcal{A}$ be a partition of (L, s). The relative entropy $D_{\mathcal{A}}(s_1 \parallel s_2)$ is convex in the pair (s_1, s_2), i.e. if $(s_1^{'}, s_2^{'})$, $(s_1^{''}, s_2^{''})$ are pairs of states on L, then*

$$D_{\mathcal{A}}((\alpha s_1^{'} + (1-\alpha)s_1^{''}) \parallel (\alpha s_2^{'} + (1-\alpha)s_2^{''})) \leq \alpha D_{\mathcal{A}}(s_1^{'} \parallel s_2^{'}) + (1-\alpha)D_{\mathcal{A}}(s_1^{''} \parallel s_2^{''}),$$

for all α with $\alpha \in [0, 1]$.

Proof. Fix $i \in \{1, 2, \dots, n\}$. Putting $x_1 = \alpha s_1^{'}(a_i)$, $x_2 = (1-\alpha)s_1^{''}(a_i)$, $y_1 = \alpha s_2^{'}(a_i)$, and $y_2 = (1-\alpha)s_2^{''}(a_i)$ in the log sum inequality, we have

$$(\alpha s_1^{'}(a_i) + (1-\alpha)s_1^{''}(a_i)) \log \frac{\alpha s_1^{'}(a_i) + (1-\alpha)s_1^{''}(a_i)}{\alpha s_2^{'}(a_i) + (1-\alpha)s_2^{''}(a_i)}$$

$$\leq \alpha p_1^{'}(a_i) \log \frac{\alpha s_1^{'}(a_i)}{\alpha s_2^{'}(a_i)} + (1-\alpha)s_1^{''}(a_i) \log \frac{(1-\alpha)s_1^{''}(a_i)}{(1-\alpha)s_2^{''}(a_i)}.$$

Summing these inequalities over all i, the result follows.

Definition 3.7 *Let $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$ and $\mathcal{B} = \{b_1, b_2, \dots, b_m\}$ be partitions of (L, s). Define mutual information as*

$$I(\mathcal{A} : \mathcal{B}) := \sum_{j=1}^m \sum_{i=1}^n s(a_i \wedge b_j) \log \frac{s(a_i \wedge b_j)}{s(a_i)s(b_j)}.$$

Notice that $I(\mathcal{A} : \mathcal{B}) = I(\mathcal{B} : \mathcal{A})$. Also, if $\mathcal{A}$ and $\mathcal{B}$ are independent, then $I(\mathcal{A} : \mathcal{B}) = 0$.

Theorem 3.8 *Let $\mathcal{A}$ and $\mathcal{B}$ be partitions of a quantum space (L, s). Then*

$$I(\mathcal{A} : \mathcal{B}) = H_s(\mathcal{A}) - H_s(\mathcal{A} \mid \mathcal{B}) = H_s(\mathcal{A}) + H_s(\mathcal{B}) - H_s(\mathcal{A} \vee \mathcal{B}).$$

Consequently, $I(\mathcal{A} : \mathcal{B}) \geq 0$, and $I(\mathcal{A} : \mathcal{A}) = 0$,

Proof. By Theorem 3.1(3), we have

$$\begin{aligned} I(\mathcal{A} : \mathcal{B}) &= \sum_{j=1}^m \sum_{i=1}^n s(a_i \wedge b_j) \log \frac{s(a_i \wedge b_j)}{s(a_i)s(b_j)} \\ &= \sum_{j=1}^m \sum_{i=1}^n s(a_i \wedge b_j) \log \frac{s(a_i \wedge b_j)}{s(b_j)} - \sum_{j=1}^m \sum_{i=1}^n s(a_i \wedge b_j) \log s(a_i) \end{aligned}$$

$$
\begin{aligned}
&= -H_s(\mathcal{A} \mid \mathcal{B}) - \sum_{j=1}^{m}\sum_{i=1}^{n} s(a_i \wedge b_j)\log s(a_i)\\
&= -H_s(\mathcal{A} \mid \mathcal{B}) - \sum_{i=1}^{n} s(a_i)\log s(a_i)\\
&= H_s(\mathcal{A}) - H_s(\mathcal{A} \mid \mathcal{B}).\\
&= H_s(\mathcal{A}) + H_s(\mathcal{B}) - H_s(\mathcal{A} \vee \mathcal{B}).
\end{aligned}
$$

Theorem 3.9 *For partitions $\mathcal{A}$ and $\mathcal{B}$ of a quantum space (L, s), we have $I(\mathcal{A} \vee \mathcal{B} : \mathcal{C}) \geq I(\mathcal{A} : \mathcal{C})$.*

Proof. By Theorem 3.1(3), (4) and Theorem 3.8, we have $I(\mathcal{A} \vee \mathcal{B} : \mathcal{C}) = H_s(\mathcal{A} \vee \mathcal{B}) - H_s(\mathcal{A} \vee \mathcal{B} \mid \mathcal{C}) = H_s(\mathcal{A}) + H_s(\mathcal{B} \mid \mathcal{A}) - H_s(\mathcal{A} \mid \mathcal{C}) - H_s(\mathcal{B} \mid \mathcal{A} \vee \mathcal{C}) = I(\mathcal{A} : \mathcal{C}) + H_s(\mathcal{B} \mid \mathcal{A}) - H_s(\mathcal{B} \mid \mathcal{A} \vee \mathcal{C}) \geq I(\mathcal{A} : \mathcal{C})$.

Definition 3.10 *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be partitions of a quantum space (L, s). The conditional mutual information of $\mathcal{A}$ and $\mathcal{B}$ given $\mathcal{C}$ is defined by*

$$I(\mathcal{A} : \mathcal{B} \mid \mathcal{C}) := H_s(\mathcal{A} \mid \mathcal{C}) - H_s(\mathcal{A} \mid (\mathcal{B} \vee \mathcal{C})).$$

Theorem 3.11 (*Chain rule for mutual information*) *If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ $(n \in N)$, and $\mathcal{B}$ are partitions of a quantum space (L, s), then*

$$I(\bigvee_{i=1}^{n} \mathcal{A}_i : \mathcal{B}) = \sum_{i=1}^{n} I(\mathcal{A}_i : \mathcal{B} \mid \bigvee_{k=1}^{i-1} \mathcal{A}_k).$$

Proof. By Theorem 3.3 and Theorem 3.8, we have

$$
\begin{aligned}
I(\bigvee_{i=1}^{n} \mathcal{A}_i : \mathcal{B}) &= H_s(\bigvee_{i=1}^{n} \mathcal{A}_i) - H_s(\bigvee_{i=1}^{n} \mathcal{A}_i \mid \mathcal{B})\\
&= \sum_{i=1}^{n} H_s(\mathcal{A}_i \mid \bigvee_{k=1}^{i-1} \mathcal{A}_k) - \sum_{i=1}^{n} H_s(\mathcal{A}_i \mid \bigvee_{k=1}^{i-1} \mathcal{A}_k \vee \mathcal{B})\\
&= \sum_{i=1}^{n} \Big(H_s(\mathcal{A}_i \mid \bigvee_{k=1}^{i-1} \mathcal{A}_k) - H_s(\mathcal{A}_i \mid \bigvee_{k=1}^{i-1} \mathcal{A}_k \vee \mathcal{B})\Big)\\
&= \sum_{i=1}^{n} I(\mathcal{A}_i : \mathcal{B} \mid \bigvee_{k=1}^{i-1} \mathcal{A}_k).
\end{aligned}
$$

Definition 3.12 *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be partitions of (L, s). Then $\mathcal{A}$ is called* conditionally independent to $\mathcal{B}$ given $\mathcal{C}$ (*written as $\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}$*) *if $\mathcal{I}(\mathcal{A} : \mathcal{C} | \mathcal{B}) = 0$.*

Theorem 3.13 $\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \iff \mathcal{C} \longrightarrow \mathcal{B} \longrightarrow \mathcal{A}$.

Proof. Let $\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}$. Then $0 = I(\mathcal{A} : \mathcal{C}|\mathcal{B}) = H_s(\mathcal{A}|\mathcal{B}) - H_s(\mathcal{A}|\mathcal{B} \vee \mathcal{C})$. Then by Theorem 3.1(3) we have, $H_s\mathcal{A}|\mathcal{B} = H_s(\mathcal{A}|\mathcal{B} \vee \mathcal{C}) = H_s(\mathcal{A} \vee \mathcal{B} \vee \mathcal{A}) - H_s(\mathcal{B} \vee \mathcal{C})$.

Now, again by Theorem 3.1(3), $I(\mathcal{C} : \mathcal{A}|\mathcal{B}) = H_s(\mathcal{C}|\mathcal{B}) - H_s(\mathcal{C}|\mathcal{A} \vee \mathcal{B}) = H_s(\mathcal{B} \vee \mathcal{C}) - H_s(\mathcal{B}) - H_s(\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) + H_s(\mathcal{B} \vee \mathcal{C}) = H_s(\mathcal{A} \vee \mathcal{B}) - H_s(\mathcal{B}) - H_s(\mathcal{A}|\mathcal{B}) = 0$.

Remark. In view of the above theorem we may write $\mathcal{A} \longleftrightarrow \mathcal{B} \longleftrightarrow \mathcal{C}$ for $\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}$ and we may say that $\mathcal{A}$ *and* $\mathcal{C}$ *are conditionally independent given* $\mathcal{B}$.

Theorem 3.14 *For any partition $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of (L, s), we have $I(\mathcal{A} : \mathcal{B} \vee \mathcal{C}) = I(\mathcal{A} : \mathcal{B}) + I(\mathcal{A} : \mathcal{C}|\mathcal{B}) = I(\mathcal{A} : \mathcal{C}) + I(\mathcal{A} : \mathcal{B}|\mathcal{C})$.*

Proof. By Theorem 3.8 we get, $I(\mathcal{A} : \mathcal{B}) + I(\mathcal{A} : \mathcal{C}|\mathcal{B}) = H_s\mathcal{A} - H_s(\mathcal{A}|\mathcal{B}) + H_s(\mathcal{A}|\mathcal{B}) - H_s(\mathcal{A}|\mathcal{B} \vee \mathcal{C}) = H_s(\mathcal{A}) - H_s(\mathcal{A}|\mathcal{B} \vee \mathcal{C}) = I(\mathcal{A} : \mathcal{B} \vee \mathcal{C})$.

Theorem 3.15 *Let $\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}$. Then*
(i) $I(\mathcal{A} \vee \mathcal{B} : \mathcal{C}) = I(\mathcal{B} : \mathcal{C})$;
(ii) $I(\mathcal{B} : \mathcal{C}) = I(\mathcal{A} : \mathcal{C}) + I(\mathcal{C} : \mathcal{B} \mid \mathcal{A})$;
(iii) $I(\mathcal{A} : \mathcal{B}|\mathcal{C}) \leq I(\mathcal{A} : \mathcal{B})$. (*Data Processing Inequality*)

Proof. (i) Let $\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}$, i.e. $I(\mathcal{A} : \mathcal{C}|\mathcal{B}) = 0$. So, by the chain rule for mutual information, we have $I(\mathcal{A} \vee \mathcal{B} : \mathcal{C}) = I(\mathcal{B} \vee \mathcal{A} : \mathcal{C}) = I(\mathcal{B} : \mathcal{C}) + I(\mathcal{A} : \mathcal{C}|\mathcal{B}) = I(\mathcal{B} : \mathcal{C})$.

(ii) By Theorem 3.14, we have

$$I(\mathcal{A} \vee \mathcal{B} : \mathcal{C}) = I(\mathcal{A} : \mathcal{C}) + I(\mathcal{C} : \mathcal{B}|\mathcal{A}).$$

Using (i), it follows that $I(\mathcal{B} : \mathcal{C}) = I(\mathcal{A} : \mathcal{C}) + I(\mathcal{C} : \mathcal{B} \mid \mathcal{A})$.

(iii) It follows from (ii) that, $I(\mathcal{C} : \mathcal{B}|\mathcal{A}) \leq I(\mathcal{B} : \mathcal{C}) = I(\mathcal{C} : \mathcal{B})$. In view of Theorem 3.13, interchanging $\mathcal{A}$ and $\mathcal{C}$, we get $I(\mathcal{A} : \mathcal{B}|\mathcal{C}) \leq I(\mathcal{A} : \mathcal{B})$.

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References

[1] B. De Baets and R. Mesiar, Fuzzy partitions and their entropy, In: *Proc. IPMU*, (1996), 1419-1424.

[2] L. Beran, *Orthomodular Lattices, Algebraic Approach*, D. Reidel, Holland, 1984.

[3] G. Birkhoff and J. Von Neumann, The logic of quantum mechanics, *Ann. Math.*, **37** (1936), 823-834.

[4] R. S. Ellis, Entropy, Large Deviations, and Statistical Mechanics, Springer-Verlag, New York, 1985.

[5] G. Kalmbach, *Ortomodular Lattices*, Academic Press, London, 1983.

[6] M. Khare and S. Roy, Conditional entropy and the Rokhlin metric on an orthomodular lattice with Bayessian state, *Int. J. Theor. Phys.*, **47**(5) (2008) 1386-1396.

[7] M. Khare and S. Roy, Entropy of quantum dynamical systems and sufficient families in orthomodular lattice with Bayessian state, *Commun. Theor. Phys.*, **50** (2008) 551-556.

[8] E. L. Marsden, The commutator and solvability in a generalized orthomoduar lattice, *Pacific J. of Math.*, **33** (1970), 457-461.

[9] M. Matoušek, Orthomoduar lattices with fully nontrivial commutators, *Comment. Math. Univ. Carolinae.*, **33** (1990), 25-32.

[10] P. Pták and S. Pulmannová, *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht, 1981.

[11] S. Pulmannová, Hidden variables and Bell inequalities on quantum logic, *Found. Phys.*, **32** (2002), 193-216.

[12] S. Pulmannová and V. Majerník, Bell inequalities on quantum logics, *J. Math. Phys.*, **33** (1992), 2173-2178.

[13] J. Pykacz, Fuzzy quantum logics and infinite-valued Lukasiewicez logic, *Inter. J. of Theo. Phy.*, **33** (1994), 1403-1416.

[14] Hong Qian, Relative entropy: free energy associated with equilibrium fluctuations and nonequilibrium deviations, arXiv:math-ph/0007010v2.

[15] H. Risken, The Fokker-Planck Equation, Methods of Solution and Applications, Springer-Verlag, New York, 1984.

[16] J. Schnakenberg, Network theory of microscopic and macroscopic behavior of master equation systems, Rev. Mod. Phys. **48** (1976), 571-585.

[17] H. Yuan, Entropy of partitions on quantum logic, *Commun. Theor. Phys.*, **43** (2005), 437-439.

[18] Z. Yue-Xu and M. Zhi-Hao, Conditional entropy of partitions on quantum logic, *Commun. Theor. Phys.*, **48** (2007), 11-13.