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A New Approach to the Study of Extended Metric Spaces

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Abstract

This paper studies the category theoretic aspects of ε-approach nearness spaces and ε-approach merotopic spaces, respectively, $\varepsilon \in (0, \infty]$, which are useful in measuring the degree of nearness (resemblance) of objects. Such structures measure the almost nearness of two collections of subsets of a nonempty set. The categories $\boldsymbol{\varepsilon ANear}$ and $\boldsymbol{\varepsilon AMer}$ are shown to be full supercategories of various well-known categories, including the category $\boldsymbol{sTop}$ of symmetric topological spaces and continuous maps, and the category $\boldsymbol{Met^{\infty}}$ of extended metric spaces and nonexpansive maps. The results in this paper have important practical implications in the study of patterns in similar pictures.

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1 Introduction

In [13], the two-argument ε-approach nearness was axiomatized for the purpose of presenting an alternative approach for completing extended metric spaces. This completion of extended metric spaces was motivated by the recent work on approach merotopic spaces in [7, 6, 4, 5, 16], (see also [2, 3]). In [15],

ε-approach nearness was used to prove Niemytzki-Tychonoff theorem for symmetric topological spaces. The notion of distance in approach spaces is closely related to the notion of nearness. The question "*how near*" the objects may be, was unsolved untill, in [5], a generalization of proximity, called approach merotopic structures were introduced which measure the degree of nearness of a collection of sets (see also [7, 4, 6]). An approach merotopic structure is a function $\nu : \mathcal{P}^2(X) \longrightarrow [0, \infty]$, and has only one argument. For comparision, we require atleast two objects, so this mainly motivates the introduction of a two-argument ε-approach merotopic structure $\nu : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$, to measure the degree of nearness between two digital images which can be considered as cllections of subimages. Further, using the notion defined in [8], two images $\mathcal{I}1$ and $\mathcal{I}2$ are either similar ($\nu(\mathcal{I}1, \mathcal{I}2) = 0$) or dissimilar ($\nu(\mathcal{I}1, \mathcal{I}2) \neq 0$). Most of the time, when we are comparing two images, they are not exactly similar (near), instead they are almost near, for example they can have different shades of same color. Therefore, a function measuring *almost nearness* of digital images was required. This problem is solved by the positive real number ε associated with an ε-approach merotopy. The images $\mathcal{I}1$ and $\mathcal{I}2$ are said to be ε-near (or *sufficiently near*) if $\nu(\mathcal{I}1, \mathcal{I}2) < \varepsilon$, where the choice of ε is in our hand. This new structure facilitates the study of the degree of nearness between nonempty disjoint collections of sets (see also [10, 12, 14, 17]). From a practical point of view, it is helpful to consider the degree of nearnes of disjoint collections of subsets such as those found in regions-of-interest in pairs of digital images (see [11]).

The focus of this paper is on the properties of the categories $\boldsymbol{\varepsilon AMer}$ and $\boldsymbol{\varepsilon ANear}$ whose objects are ε-approach merotopic spaces and ε-approach nearness spaces, respectively. The notation $\boldsymbol{A} \hookrightarrow \boldsymbol{B}$ reads *category* $\boldsymbol{A}$ *is embedded in category* $\boldsymbol{B}$. The categories $\boldsymbol{\varepsilon AMer}$ and $\boldsymbol{\varepsilon ANear}$ are supercategories for a variety of familiar categories shown in Fig. 1. Let $\boldsymbol{\varepsilon ANear}$ denote the category of all ε-approach nearness spaces and contractions, and let $\boldsymbol{\varepsilon AMer}$ denote the category of all ε-approach merotopic spaces and contractions.

Among these familiar categories is $\boldsymbol{sTop}$, the symmetric form of $\boldsymbol{Top}$, the category with objects that are topological spaces and morphisms that are continuous maps between them [1, 9].

Again, for example, $\boldsymbol{Met^\infty}$ with objects that are extended metric spaces is a subcategory of $\boldsymbol{\varepsilon AP}$ (having objects ε-approach spaces and contractions) (see also [12, 16]). The map $f : (X, \rho_X) \longrightarrow (Y, \rho_Y)$ is a contraction if and only if $f : (X, \nu_{D_{\rho_X}}) \longrightarrow (Y, \nu_{D_{\rho_Y}})$ is a contraction. Thus $\varepsilon\mathbf{AP}$ is embedded as a

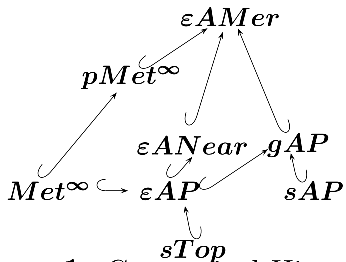

Figure 1: Categorical Hierarchy

full subcategory in $\boldsymbol{\varepsilon ANear}$ by the functor $F : \boldsymbol{\varepsilon AP} \longrightarrow \boldsymbol{\varepsilon ANear}$ defined by $F((X,\rho)) = (X, \nu_{D_\rho})$ and $F(f) = f$. Then $f : (X, \rho_X) \longrightarrow (Y, \rho_Y)$ is a contraction if and only if $f : (X, \nu_{D_{\rho_X}}) \longrightarrow (Y, \nu_{D_{\rho_Y}})$ is a contraction. Thus $\boldsymbol{\varepsilon AP}$ is embedded as a full subcategory in $\boldsymbol{\varepsilon ANear}$ by the functor $F : \boldsymbol{\varepsilon AP} \longrightarrow \boldsymbol{\varepsilon ANear}$ defined by $F((X,\rho)) = (X, \nu_{D_\rho})$ and $F(f) = f$. Since the category $\boldsymbol{Met^\infty}$ of extended metric spaces and nonexpansive maps is a full subcategory of $\boldsymbol{\varepsilon AP}$, therefore, $\boldsymbol{\varepsilon ANear}$ is also a full supercategory of $\boldsymbol{Met^\infty}$. Fig. 1 is the pictorial representation of the hierarchy of categories which is a consequence of the study done in this paper.

2 Preliminaries

In this section, generalized approach spaces are briefly discussed. A function $\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to [0,\infty]$ is a distance on X, provided, for all nonempty $A, B, C \in \mathcal{P}(X)$,

(**D**.1) $\delta(A, A) = 0$,
(**D**.2) $\delta(A, \emptyset) = \infty$,
(**D**.3) $\delta(A, B \cup C) = \min\{\delta(A,B), \delta(A,C)\}$,
(**D**.4) $\delta(A,B) \le \delta(A, B^{(\alpha)}) + \alpha$, for $\alpha \in [0,\infty]$, where $B^{(\alpha)} \doteqdot \{x \in X : \delta(\{x\}, B) \le \alpha\}$.

The distance δ first appeared in [13], an extension of the distance in [8]. The pair (X,δ) is called a *generalized approach space.* This leads to a new form of distance called an ε-approach merotopy.

The following function called *gap functional* was introduced by Čech in his 1936–1939 seminar on topology [18]:

Example 2.1 *For nonempty subsets $A, B \in \mathcal{P}(X)$, the distance function $D_\rho : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow [0,\infty]$ is defined by*

$$D_\rho(A,B) = \begin{cases} \inf\{\rho(a,b) : a \in A, b \in B\}, & \text{if } A \text{ and } B \text{ are not empty,} \\ \infty, & \text{if } A \text{ or } B \text{ is empty.} \end{cases}$$

Observe that (X, D_ρ) is a generalized approach space, where ρ is an extended psuedo-metric on X.

3 The Categories $\boldsymbol{\varepsilon ANear}$ and $\boldsymbol{\varepsilon AMer}$

In this section, we study the categories $\boldsymbol{\varepsilon ANear}$ and $\boldsymbol{\varepsilon AMer}$ having objects ε-approach nearness spaces and ε-approach merotopic spaces, respectively. Various known topological categories are shown to be full subcategories of these categories. Let

$$\mathcal{A} \vee \mathcal{B} \doteqdot \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\},$$
$$\mathcal{A} \prec \mathcal{B} \Leftrightarrow \forall A \in \mathcal{A}, \exists B \in \mathcal{B} : B \subseteq A \textit{ i.e.}, \mathcal{A} \text{ corefines } \mathcal{B}.$$

Definition 3.1 *Let $\varepsilon \in (0, \infty]$. Then a function $\nu : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ is an ε-approach merotopy on X if and only if for any collections $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{P}^2(X)$, the properties (AN.1)-(AN.5) are satisfied:*

(AN.1) $\mathcal{A} \prec \mathcal{B} \Longrightarrow \nu(\mathcal{C}, \mathcal{A}) \leq \nu(\mathcal{C}, \mathcal{B})$,

(AN.2) $\mathcal{A} \neq \emptyset, \mathcal{B} \neq \emptyset$ *and* $(\bigcap \mathcal{A}) \cap (\bigcap \mathcal{B}) \neq \emptyset \Longrightarrow \nu(\mathcal{A}, \mathcal{B}) < \varepsilon$,

(AN.3) $\nu(\mathcal{A}, \mathcal{B}) = \nu(\mathcal{B}, \mathcal{A})$ *and* $\nu(\mathcal{A}, \mathcal{A}) = 0$,

(AN.4) $\mathcal{A} \neq \emptyset \Longrightarrow \nu(\emptyset, \mathcal{A}) = \infty$,

(AN.5) $\nu(\mathcal{C}, \mathcal{A} \vee \mathcal{B}) \geq \nu(\mathcal{C}, \mathcal{A}) \wedge \nu(\mathcal{C}, \mathcal{B})$.

The pair (X, ν) is termed as an ε-approach merotopic space.

For an ε-approach merotopic space (X, ν), we define: $cl_\nu(A) \doteqdot \{x \in X : \nu(\{\{x\}\}, \{A\}) < \varepsilon\}$, for all $A \subseteq X$. Then cl_ν is a Čech closure operator on X.

Let $cl_\nu(\mathcal{A}) \doteqdot \{cl_\nu(A) : A \in \mathcal{A}\}$. Then an ε-approach merotopy ν on X is called an *ε-approach nearness* on X, if the following condition is satisfied:

(AN.6) $\nu(cl_\nu(\mathcal{A}), cl_\nu(\mathcal{B})) \geq \nu(\mathcal{A}, \mathcal{B})$.

In this case, cl_ν is a Kuratowski closure operator on X.

For an ε-approach nearness ν that satisfies (AN.6), (X, ν) is an *ε-approach nearness space*. For a source of examples of ε-approach nearness on a nonempty set X, consider the following example:

Example 3.2 *Let D_ρ be a gap functional. Then the function $\nu_{D_\rho} : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ defined as*

$$\nu_{D_\rho}(\mathcal{A}, \mathcal{B}) \doteqdot \sup_{A \in \mathcal{A}, B \in \mathcal{B}} D_\rho(A, B); \quad \nu_{D_\rho}(\mathcal{A}, \mathcal{A}) \doteqdot \sup_{A \in \mathcal{A}} D_\rho(A, A) = 0,$$

*is an ε-approach merotopy on X. Define $cl_\rho(A) = \{x \in X : \rho(\{x\}, A) < \varepsilon\}$, $A \subseteq X$. Then cl_ρ is a Čech closure operator on X. Further, if $\rho(cl_\rho(A), cl_\rho(B)) \geq \rho(A, B)$, for all $A, B \subseteq X$, then cl_ρ is a Kuratowski closure operator on X, and we call ρ as an ε-*approach function *on X; and (X, ρ) is an ε-*approach space*. In this case, ν_{D_ρ} is an ε-approach nearness on X.*

So, there are many instances of ε-approach nearness on X just as there are many instances of ε-approach spaces [8] and metric spaces on X.

Example 3.3 *Let $\varepsilon \in (0, \infty]$. Then the function $\nu_d : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ defined as: for $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$, $\nu_d(\mathcal{A}, \mathcal{B}) = 0$, if ($\mathcal{A}$ and $\mathcal{B}$ are nonempty collections and $(\bigcap \mathcal{A}) \cap (\bigcap \mathcal{B}) \neq \emptyset$) or $\mathcal{A} = \mathcal{B}$, and $\nu_d(\mathcal{A}, \mathcal{B}) = \infty$, otherwise, is an ε-approach nearness on X and $cl_{\nu_d}(A) = A$, for all $A \subseteq X$. We call (X, ν_d) a* discrete ε-approach nearness space. *Further, the function $\nu_i : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ defined as: for $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$, $\nu_i(\mathcal{A}, \mathcal{B}) = 0$, if ($\mathcal{A} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$) or $\mathcal{A} = \mathcal{B}$, and $\nu_i(\mathcal{A}, \mathcal{B}) = \infty$, otherwise, is an ε-approach nearness on X and $cl_{\nu_i}(A) = X$, for all nonempty subsets A of X. We call (X, ν_i) an* indiscrete ε-approach nearness space.

In the following set of examples, we develop methods of obtaining a new ε-approach nearness from a given ε-approach nearness on X.

Example 3.4 *Let (X, ν) be an ε-approach nearness on X, $\varepsilon < r < \infty$ and $\varepsilon' < \varepsilon$. Then*

1. $\nu_1 : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ *defined by*

$$\nu_1(\mathcal{A}, \mathcal{B}) = \begin{cases} \infty, & \text{if } \mathcal{A} = \emptyset \text{ or } \mathcal{B} = \emptyset, \\ \nu(\mathcal{A}, \mathcal{B}) \wedge r, & \text{otherwise,} \end{cases}$$

is an ε-approach nearness on X.

2. $\nu_2 : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ *defined by*

$$\nu_2(\mathcal{A}, \mathcal{B}) = \begin{cases} \nu(\mathcal{A}, \mathcal{B}) \wedge \varepsilon', & \text{if } \nu(\mathcal{A}, \mathcal{B}) < \varepsilon, \\ \nu(\mathcal{A}, \mathcal{B}) \vee r, & \text{otherwise,} \end{cases}$$

is an ε-approach nearness on X.

3. $\nu_3 : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ *defined by*

$$\nu_3(\mathcal{A}, \mathcal{B}) = \sup\{\nu(\mathcal{C}, \mathcal{D}) : \mathcal{C} \subseteq \mathcal{A} \text{ and } \mathcal{D} \subseteq \mathcal{B}, \text{ such that } 0 < |\mathcal{C}| < \aleph_0 \text{ and } 0 < |\mathcal{D}| < \aleph_0\}$$

and

$$\nu_3(\mathcal{A}, \mathcal{A}) = \sup\{\nu(\mathcal{C}, \mathcal{C}) : \mathcal{C} \subseteq \mathcal{A} \text{ such that } |\mathcal{C}| < \aleph_0\} = 0,$$

is an ε-approach nearness on X. Here $\aleph_0$ is the first infinite cardinal number.

Having established an adequate number of examples of ε-approach nearness spaces, it is now relevant to study the category $\boldsymbol{\varepsilon ANear}$. For this, we require the following definition:

Definition 3.5 *For any ε-approach nearness spaces (X,ν) and (Y,ν'), a map $f : X \longrightarrow Y$ is called a contraction if $\nu'(f(\mathcal{A}), f(\mathcal{B})) \leq \nu(\mathcal{A},\mathcal{B})$, for all $\mathcal{A},\mathcal{B} \in \mathcal{P}^2(X)$.*

Lemma 3.6 *Let $\varepsilon \in (0,\infty]$, and let (X,ν) and (Y,ν') be ε-approach nearness spaces. Then $f : (X,\nu) \longrightarrow (Y,\nu')$ is a contraction if and only if $\nu(f^{-1}(\mathcal{A}), f^{-1}(\mathcal{B})) \geq \nu'(\mathcal{A},\mathcal{B})$, for all $\mathcal{A},\mathcal{B} \in \mathcal{P}^2(Y)$.*

Remark 3.7 *Let $\boldsymbol{\varepsilon ANear}$ denote the category of ε-approach nearness spaces and contractions, and $\boldsymbol{\varepsilon AP}$ denote the category of ε-approach spaces and contractions. Suppose that (X,ρ_X) and (Y,ρ_Y) are ε-approach spaces. Then $f : (X,\rho_X) \longrightarrow (Y,\rho_Y)$ is a contraction if and only if $f : (X,\nu_{D_{\rho_X}}) \longrightarrow (Y,\nu_{D_{\rho_Y}})$ is a contraction. Thus $\boldsymbol{\varepsilon AP}$ is embedded as a full subcategory in $\boldsymbol{\varepsilon ANear}$ by the functor $F : \boldsymbol{\varepsilon AP} \longrightarrow \boldsymbol{\varepsilon ANear}$ defined as: $F((X,\rho)) = (X,\nu_{D_\rho})$ and $F(f) = f$. Since the category $\boldsymbol{Met^\infty}$ of extended metric spaces and nonexpansive maps is a full subcategory of $\boldsymbol{\varepsilon AP}$, therefore, $\boldsymbol{\varepsilon ANear}$ is also a full supercategory of $\boldsymbol{Met^\infty}$.*

Theorem 3.8 *Let $\varepsilon \in (0,\infty]$. Then the category $\boldsymbol{\varepsilon ANear}$ is a topological construct.*

Proof. Clearly, the category $\boldsymbol{\varepsilon ANear}$ is concrete. Let $((X_j,\nu_j))_{j\in J}$ be a family of ε-approach nearness spaces and let $(f_j : X \longrightarrow X_j)_{j\in J}$ be a source in $\boldsymbol{\varepsilon ANear}$. Define $\nu : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0,\infty]$ by

$$\nu(\mathcal{A},\mathcal{B}) \doteqdot \sup\{\inf_{i,k=1}^{m,n} \sup_{j\in J} \nu_j(f_j(\mathcal{A}_i), f_j(\mathcal{B}_k)) : (\mathcal{A}_i)_{i=1}^m \in \mathfrak{C}(\mathcal{A}) \text{ and } (\mathcal{B}_k)_{k=1}^n \in \mathfrak{C}(\mathcal{B})\},$$

for every $\mathcal{A},\mathcal{B} \in \mathcal{P}^2(X)$, where $\mathfrak{C}(\mathcal{A}) = \{(\mathcal{A}_i)_{i=1}^m : \mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_m \prec \mathcal{A},\ m \in \mathbb{N}\}$ and $\mathfrak{C}(\mathcal{B}) = \{(\mathcal{B}_i)_{i=1}^n : \mathcal{B}_1 \vee \mathcal{B}_2 \vee \cdots \vee \mathcal{B}_n \prec \mathcal{B},\ n \in \mathbb{N}\}$; here $\mathbb{N}$ denotes the set of natural numbers. Then we will show that ν is the initial ε-approach nearness on X. For this, we will first show that ν is an ε-approach nearness on X. (AN.1) is obvious. Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty collections and $(\bigcap\mathcal{A}) \cap (\bigcap\mathcal{B}) \neq \emptyset$. Then there exists $x \in \bigcap\mathcal{A}$ such that $x \in \bigcap\mathcal{B}$, for some $x \in X$. If $(\mathcal{A}_i)_{i=1}^m \in \mathfrak{C}(\mathcal{A})$ and $(\mathcal{B}_k)_{k=1}^n \in \mathfrak{C}(\mathcal{B})$, then

$$\nu_j(\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_m, \mathcal{B}_1 \vee \mathcal{B}_2 \vee \cdots \vee \mathcal{B}_n) < \varepsilon, \text{ for all } j \in J,$$

i.e., for all $j \in J$, we have

$$\nu_j(\mathcal{A}_1, \mathcal{B}_1 \vee \mathcal{B}_2 \vee \cdots \vee \mathcal{B}_n) \wedge \nu_j(\mathcal{A}_2, \mathcal{B}_1 \vee \mathcal{B}_2 \vee \cdots \vee \mathcal{B}_n) \wedge \cdots \wedge \nu_j(\mathcal{A}_m, \mathcal{B}_1 \vee \mathcal{B}_2 \vee \cdots \vee \mathcal{B}_n) < \varepsilon$$

$\Longrightarrow \nu_j(\mathcal{A}_1,\mathcal{B}_1)\wedge\nu_j(\mathcal{A}_1,\mathcal{B}_2)\wedge\cdots\wedge\nu_j(\mathcal{A}_1,\mathcal{B}_n)\wedge\cdots\wedge\nu_j(\mathcal{A}_m,\mathcal{B}_n) < \varepsilon.$

Thus, for each $j \in J$, there exists $i \in \{1,2,\ldots,m\}$ and $k \in \{1,2,\ldots,n\}$ such that $\nu_j(\mathcal{A}_i,\mathcal{B}_k) < \varepsilon$. Since each ν_j is a contraction, therefore $\nu_j(f_j(\mathcal{A}_i), f_j(\mathcal{B}_k)) < \varepsilon$, for some $i \in \{1,2,\ldots,m\}$ and $k \in \{1,2,\ldots,n\}$ and $j \in J$. Consequently, (AN.2) follows. Clearly, $\nu(\mathcal{A},\mathcal{B}) = \nu(\mathcal{B},\mathcal{A})$. Let $\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n \prec \mathcal{A}$ and $\mathcal{B}_1\vee\mathcal{B}_2\vee\cdots\vee\mathcal{B}_m \prec \mathcal{A}$. Then $\nu_j(\mathcal{A}_1\vee\mathcal{A}_2\vee\cdots\vee\mathcal{A}_n, \mathcal{B}_1\vee\mathcal{B}_2\vee\cdots\vee\mathcal{B}_m) \leq \nu_j(\mathcal{A},\mathcal{A}) = 0$, for all $j \in J$. That is, for all $j \in J$, there exists $i \in \{1,2,\ldots,n\}$ and $k \in \{1,2,\ldots,m\}$ such that $\nu_j(\mathcal{A}_i,\mathcal{B}_k) = 0$ which gives that $\nu_j(f_j(\mathcal{A}_i), f_j(\mathcal{B}_k)) = 0$. Thus $\nu(\mathcal{A},\mathcal{A}) = 0$. (AN.4) is obvious. For (AN.5), let $(\mathcal{A}_i)_{i=1}^n \in \mathfrak{C}(\mathcal{B}_1)$ and $(\mathcal{D}_i)_{i=1}^m \in \mathfrak{C}(\mathcal{B}_2)$. Then $\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n \vee \mathcal{D}_1 \vee \mathcal{D}_2 \vee \cdots \vee \mathcal{D}_m \prec \mathcal{B}_1 \vee \mathcal{B}_2$. Thus $(\mathcal{A}_i)_{i=1}^n \cup (\mathcal{D}_i)_{i=1}^m \in \mathfrak{C}(\mathcal{B}_1 \vee \mathcal{B}_2)$. As a result, $\mathfrak{C}(\mathcal{B}_1 \vee \mathcal{B}_2) = \mathfrak{C}(\mathcal{B}_1) \vee \mathfrak{C}(\mathcal{B}_2)$. Hence $\nu(\mathcal{A},\mathcal{B}_1 \vee \mathcal{B}_2) = \nu(\mathcal{A},\mathcal{B}_1) \wedge \nu(\mathcal{A},\mathcal{B}_2)$. Finally,

$$\begin{aligned}
&\nu(cl_\nu(\mathcal{A}), cl_\nu(\mathcal{B}))\\
&= \sup\{\inf_{i,k=1}^{m,n}\sup_{j\in J}\nu_j(f_j(\mathcal{A}_i), f_j(\mathcal{B}_k)) : (\mathcal{A}_i)_{i=1}^m \in \mathfrak{C}(cl_\nu(\mathcal{A})) \text{ and } (\mathcal{B}_k)_{k=1}^n \in \mathfrak{C}(cl_\nu(\mathcal{B}))\}\\
&\geq \sup\{\inf_{i,k=1}^{m,n}\sup_{j\in J}\nu_j(f_j(cl_\nu(\mathcal{A}_i)), f_j(cl_\nu(\mathcal{B}_k))) : (\mathcal{A}_i)_{i=1}^m \in \mathfrak{C}(cl_\nu(\mathcal{A})) \text{ and } (\mathcal{B}_k)_{k=1}^n \in \mathfrak{C}(cl_\nu(\mathcal{B}))\}\\
&\geq \sup\{\inf_{i,k=1}^{m,n}\sup_{j\in J}\nu_j(cl_{\nu_j}(f_j(\mathcal{A}_i)), cl_{\nu_j}(f_j(\mathcal{B}_k))) : (\mathcal{A}_i)_{i=1}^m \in \mathfrak{C}(cl_\nu(\mathcal{A})) \text{ and } (\mathcal{B}_k)_{k=1}^n \in \mathfrak{C}(cl_\nu(\mathcal{B}))\}\\
&= \sup\{\inf_{i,k=1}^{m,n}\sup_{j\in J}\nu_j(f_j(\mathcal{A}_i), f_j(\mathcal{B}_k)) : (\mathcal{A}_i)_{i=1}^m \in \mathfrak{C}(cl_\nu(\mathcal{A})) \text{ and } (\mathcal{B}_k)_{k=1}^n \in \mathfrak{C}(cl_\nu(\mathcal{B}))\}\\
&\geq \sup\{\inf_{i,k=1}^{m,n}\sup_{j\in J}\nu_j(f_j(\mathcal{A}_i), f_j(\mathcal{B}_k)) : (\mathcal{A}_i)_{i=1}^m \in \mathfrak{C}(\mathcal{A}) \text{ and } (\mathcal{B}_k)_{k=1}^n \in \mathfrak{C}(\mathcal{B})\}\\
&= \nu(\mathcal{A},\mathcal{B}).
\end{aligned}$$

To show that ν is the initial ε-approach nearness on X, let (Y,ν') be an ε-approach nearness space and $g : Y \longrightarrow X$ be a map such that $f_j \circ g : Y \longrightarrow X_j$ is a contraction, for each $j \in J$. Then we will show that g is a contraction. Let $\mathcal{A},\mathcal{B} \in \mathcal{P}^2(Y)$. Suppose that $\nu(g(\mathcal{A}), g(\mathcal{B})) > \nu'(\mathcal{A},\mathcal{B})$. Then, for some $j \in J$, we have

$$\begin{aligned}
\nu'(\mathcal{A},\mathcal{B}) &< \inf_{i,k=1}^{m,n}\nu_j(f_j(\mathcal{A}_i), f_j(\mathcal{B}_k)), \text{ for some } (\mathcal{A}_i)_{i=1}^m \in \mathfrak{C}(g(\mathcal{A})), (\mathcal{B}_k)_{k=1}^n \in \mathfrak{C}(g(\mathcal{B}))\\
&\leq \nu_j(f_j(\mathcal{A}_1) \vee f_j(\mathcal{A}_2) \vee \cdots \vee f_j(\mathcal{A}_m), f_j(\mathcal{B}_1) \vee f_j(\mathcal{B}_2) \vee \cdots \vee f_j(\mathcal{B}_n))\\
&< \nu_j(f_j(\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_m), f_j(\mathcal{B}_1 \vee \mathcal{B}_2 \vee \cdots \vee \mathcal{B}_n))\\
&\leq \nu_j(f_j(g(\mathcal{A})), f_j(g(\mathcal{B})))\\
&\leq \nu'(\mathcal{A},\mathcal{B}), \text{ since } f_j \circ g : Y \longrightarrow X_j \text{ is a contraction,}
\end{aligned}$$

i.e., $\nu'(\mathcal{A},\mathcal{B}) < \nu'(\mathcal{A},\mathcal{B})$, which is absurd. Therefore, $\nu'(\mathcal{A},\mathcal{B}) \geq \nu(g(\mathcal{A}), g(\mathcal{B}))$, for all $\mathcal{A},\mathcal{B} \in \mathcal{P}^2(X)$. Consequently, g is a contraction and ν is the initial ε-approach nearness on X. As a result, $\boldsymbol{\varepsilon ANear}$ is a topological construct.

Let $\varepsilon \in (0,\infty]$. Suppose that $\boldsymbol{\varepsilon AMer}$ denotes the category of all ε-approach merotopic spaces and contractions. Then we have the following result.

Corollary 3.9 *Let $\varepsilon \in (0,\infty]$. Then the category $\boldsymbol{\varepsilon AMer}$ is a topological construct.*

Proof. The proof is similar to the proof of Theorem 3.8, with the same initial structure.

The above results confirm the existence of final structures of the categories ε**ANEAR** and $\boldsymbol{\varepsilon AMer}$, $\varepsilon \in (0,\infty]$. In the following series of results, we explicitly construct the final structures of these categories.

Proposition 3.10 *Let $\varepsilon \in (0,\infty]$. For any family $((X_j,\nu_j))_{j\in J}$ of $\boldsymbol{\varepsilon ANear}$-objects and a sink $(f_j : X_j \longrightarrow X)_{j\in J}$, the function $\nu : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0,\infty]$ defined by: $\mathcal{A},\mathcal{B} \in \mathcal{P}^2(X)$,*

$$\nu(\mathcal{A},\mathcal{B}) \doteqdot \begin{cases} 0, & \text{if } (\bigcap cl(\mathcal{A})) \cap (\bigcap cl(\mathcal{B})) \neq \emptyset, \\ \inf\limits_{j\in J} \nu_j(f_j^{-1}(cl(\mathcal{A})), f_j^{-1}(cl(\mathcal{B}))), & \text{otherwise,} \end{cases}$$

where $cl(A) \doteqdot \bigcap\{B \subseteq X : A \subseteq B \text{ and } cl_{\nu_j}(B) = B, \forall j \in J\}$, is the final ε-approach nearness on X.

Proof. First we will show that ν is an ε-approach nearness on X. Clearly, ν satisfies (AN.1), (AN.2), (AN.3) and (AN.4). For (AN.5), let $\mathcal{A},\mathcal{B}_1,\mathcal{B}_2 \in \mathcal{P}^2(X)$ such that $(\bigcap cl(\mathcal{A})) \cap (\bigcap cl(\mathcal{B}_1)) = \emptyset$ and $\bigcap cl(\mathcal{A})) \cap (\bigcap cl(\mathcal{B}_2)) = \emptyset$. Then

$$\begin{aligned} \nu(\mathcal{A},\mathcal{B}_1 \vee \mathcal{B}_2) &= \inf_{j\in J} \nu_j(f_j^{-1}(cl(\mathcal{A})), f_j^{-1}(cl(\mathcal{B}_1 \vee \mathcal{B}_2))) \\ &= \inf_{j\in J} \nu_j(f_j^{-1}(cl(\mathcal{A})), f_j^{-1}(cl(\mathcal{B}_1)) \vee f_j^{-1}(cl(\mathcal{B}_2))) \\ &= \inf_{j\in J} \nu_j(f_j^{-1}(cl(\mathcal{A})), f_j^{-1}(cl(\mathcal{B}_1))) \wedge \inf_{j\in J} \nu_j(f_j^{-1}(cl(\mathcal{A})), f_j^{-1}(cl(\mathcal{B}_2))) \\ &= \nu(\mathcal{A},\mathcal{B}_1) \wedge \nu(\mathcal{A},\mathcal{B}_2). \end{aligned}$$

For (AN.6), we will only show that $cl_\nu = cl$. Let $x \notin cl_\nu(A)$, where $A \subseteq X$. Then $\nu(\{\{x\}\},\{A\}) \geq \varepsilon$ which yields that $cl(\{x\}) \cap cl(A) = \emptyset$. Consequently, $x \notin cl(A)$. Thus $cl(A) \subseteq cl_\nu(A)$. For the reverse inclusion, let $x \in cl_\nu(A)$ and $cl(\{x\}) \cap cl(A) \neq \emptyset$. Then there exists $y \in X$ such that $y \in cl(\{x\})$ and $y \in$

$cl(A)$. Since $y \in cl(\{x\}) \Longrightarrow x \in cl(\{y\})$, therefore $x \in cl(A)$. Next suppose that $cl(\{x\}) \cap cl(A) = \emptyset$ and $x \notin cl(A)$. Then there exists $B \subseteq X$ such that $A \subseteq B$ and $cl_{\nu_j}(B) = B$, for all $j \in J$ but $x \notin B$. Therefore $x \notin cl_{\nu_j}(A)$ which yields that $\nu_j(\{\{x\}\}, \{A\}) \geq \varepsilon$, for all $j \in J$. As a resultant, $\nu_j(\{cl(\{x\})\}, \{cl(A)\}) \geq \varepsilon$ which in turn gives that $\nu_j(\{f_j^{-1}(cl(\{x\}))\}, \{f_j^{-1}(cl(A))\}) \geq \varepsilon$, for all $j \in J$, that is $\nu(\{\{x\}\}, \{A\}) \geq \varepsilon$ implying that $x \notin cl_\nu(A)$. Thus $cl_\nu(A) = cl(A)$, for all $A \subseteq X$.

To show that ν is the final ε-approach nearness on X, let (Y, ν') be an ε-approach nearness space and $g : X \longrightarrow Y$ be a map such that $g \circ f_j : X \longrightarrow Y$ is a contraction for each $j \in J$. Let $A \subseteq X$. Then $f_j^{-1}(cl(A)) \in X_j$. Since each contraction is a continuous map, therefore $(g \circ f_j)(f_j^{-1}(cl(A))) \subseteq cl_{\nu'}(g \circ f_j \circ f_j^{-1}(cl(A))) \subseteq cl_{\nu'}(g(cl(A))) \subseteq cl_{\nu'} cl_{\nu'}(g(A)) = cl_{\nu'}(g(A))$ (the last inclusion follows because cl is the final closure operator on X). Thus $cl_{\nu'}(g(\mathcal{A})) \prec g \circ f_j \circ f_j^{-1}(cl(\mathcal{A}))$. Consequently, for $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$ and $(\bigcap cl(\mathcal{A})) \cap (\bigcap cl(\mathcal{A})) = \emptyset$, we have

$$\begin{aligned}\nu'(cl_{\nu'}(g(\mathcal{A})), cl_{\nu'}(g(\mathcal{B}))) &\leq \nu'(g \circ f_j \circ f_j^{-1}(cl(\mathcal{A}))), g \circ f_j \circ f_j^{-1}(cl(\mathcal{B})))) \\ &\leq \nu_j(f_j^{-1}(cl(\mathcal{A}))), f_j^{-1}(cl(\mathcal{B})))), \text{ for all } j \in J.\end{aligned}$$

Therefore

$$\nu'(cl_{\nu'}(g(\mathcal{A})), cl_{\nu'}(g(\mathcal{B}))) \leq \inf_{j \in J} \nu_j(f_j^{-1}(cl(\mathcal{A}))), f_j^{-1}(cl(\mathcal{B})))).$$

Hence

$$\begin{aligned}\nu(\mathcal{A}, \mathcal{B}) &= \inf_{j \in J} \nu_j(f_j^{-1}(cl(\mathcal{A}))), f_j^{-1}(cl(\mathcal{B})))) \\ &\geq \nu'(cl_{\nu'}(g(\mathcal{A})), cl_{\nu'}(g(\mathcal{B}))) \\ &= \nu'(g(\mathcal{A}), g(\mathcal{B})).\end{aligned}$$

Consequently, $g : (X, \nu) \longrightarrow (Y, \nu')$ is a contraction and ν is the final ε-approach nearness on X.

Corollary 3.11 *Let $\varepsilon \in (0, \infty]$. Suppose that $((X_j, \nu_j))_{j \in J}$ be a family of ε-approach merotopic spaces and $(f_j : X_j \longrightarrow X)_{j \in J}$ be a sink in $\boldsymbol{\varepsilon}$**AMer**. Then the final ε-approach merotopy on X is the function $\nu : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ defined by: $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$,*

$$\nu(\mathcal{A}, \mathcal{B}) \doteqdot \begin{cases} 0, & \textit{if } (\bigcap \mathcal{A}) \cap (\bigcap \mathcal{B}) \neq \emptyset, \\ \inf\limits_{j \in J} \nu_j(f_j^{-1}(\mathcal{A}), f_j^{-1}(\mathcal{B})), & \textit{otherwise.} \end{cases}$$

Remark 3.12 *Let (X,δ) be an approach space (as defined by Lowen [8]). Let δ also satisfies $\inf\limits_{a\in A}\delta(a,B) = \inf\limits_{b\in B}\delta(b,A)$. Then (X,δ) is a symmetric approach space. Define $\rho_\delta : \mathcal{P}(X)\times\mathcal{P}(X) \longrightarrow [0,\infty]$ by: for $A,B\subseteq X$,*

$$\rho_\delta(A,B) \doteqdot \inf_{a\in A}\delta(a,B).$$

Then ρ_δ is a generalized distance function on X. Let (X,δ_1) and (Y,δ_2) be approach spaces. Then $f : (X,\delta_1) \longrightarrow (Y,\delta_2)$ is a contraction if and only if $f : (X,\rho_{\delta_1}) \longrightarrow (Y,\rho_{\delta_2})$ is a contraction. Thus the category $\boldsymbol{sAP}$ of symmetric approach spaces and contractions is embedded as a full subcategory into the category $\boldsymbol{gAP}$ of generalized approach spaces and contractions by the functor $F : \boldsymbol{sAP} \longrightarrow \boldsymbol{gAP}$ defined as: $F((X,\delta)) = (X,\rho_\delta)$ and $F(f) = f$.

Further if (X,ρ) is a generalized approach space, then the function $\nu_\rho : \mathcal{P}^2(X)\times\mathcal{P}^2(X) \longrightarrow [0,\infty]$ defined by: for $\mathcal{A},\mathcal{B}\in\mathcal{P}^2(X)$,

$$\nu_\rho(\mathcal{A},\mathcal{B}) \doteqdot \sup_{A\in\mathcal{A},B\in\mathcal{B}}\rho(A,B) \text{ and } \nu_\rho(\mathcal{A},\mathcal{A}) \doteqdot \sup_{A\in\mathcal{A}}\rho(A,A) = 0,$$

is an ε-merotopy on X. Also for generalized approach spaces (X,ρ_1) and (Y,ρ_2), $f : (X,\rho_1) \longrightarrow (Y,\rho_2)$ is a contraction if and only if $f : (X,\nu_{\rho_1}) \longrightarrow (Y,\nu_{\rho_2})$ is a contraction. Thus the category $\boldsymbol{gAP}$ is embedded as a full subcategory into the category $\boldsymbol{\varepsilon AMer}$ by the functor $F : \boldsymbol{gAP} \longrightarrow \boldsymbol{\varepsilon AMer}$ defined as: $F((X,\rho)) = (X,\nu_\rho)$ and $F(f) = f$.

Example 3.13 *Let (X,cl) be a symmetrical topological space. Define $\delta : X\times\mathcal{P}(X) \longrightarrow [0,\infty]$ by: for $x\in X$ and $A\subseteq X$,*

$$\delta(x,A) = \begin{cases} 0, & \text{if } x\in cl(\{a\}) \text{ for some } a\in cl(A),\\ \infty, & \text{otherwise.}\end{cases}$$

Then (X,δ) is a symmetric approach space [8].

Remark 3.14 *Let (X,cl) be a symmetric topological space. Define $\delta_{cl} : X\times\mathcal{P}(X) \longrightarrow [0,\infty]$ by: for $x\in X$ and $A\subseteq X$,*

$$\delta_{cl}(x,A) \doteqdot \begin{cases} 0, & \text{if } x\in cl(A),\\ \infty, & \text{otherwise.}\end{cases}$$

Then δ_{cl} is a distance function as shown by Lowen [8]. Also $\inf\limits_{a\in cl(A)}\delta(a,B) = \inf\limits_{b\in cl(B)}\delta(b,A)$. Further define $\rho_{\delta_{cl}} : \mathcal{P}(X)\times\mathcal{P}(X) \longrightarrow [0,\infty]$ by: for $A,B\subseteq X$,

$$\rho_{\delta_{cl}}(A,B) = \inf_{a\in cl(A)}\delta_{cl}(a,B).$$

*Then $(X, \rho_{\delta_{cl}})$ is an ε-approach space and $cl_{\rho_{\delta_{cl}}} = cl = cl_{\delta_{cl}}$. Further, for symmetric topological spaces (X, cl_1) and (Y, cl_2), $f : (X, cl_1) \longrightarrow (Y, cl_2)$ is continuous if and only if $f : (X, \rho_{\delta_{cl_1}}) \longrightarrow (Y, \rho_{\delta_{cl_2}})$ is a contraction. Thus, the category **sTop** of symmetric topological spaces and continuous maps is embedded as a full subcategory into the category **εAP** of ε-approach spaces and contractions by the functor **sTop** $\longrightarrow$ **εAP** defined as: $F((X, cl)) = (X, \rho_{\delta_{cl}})$ and $F(f) = f$.*

Let us now concentrate upon the lattice theoretic properties of ε-approach merotopies and ε-approach nearness on a nonempty set X. We, hereby, construct the exact join and meet of these lattices.

Definition 3.15 *Let $\varepsilon \in (0, \infty]$. Suppose that ν and ν' be ε-approach merotopies on X. Then $\nu' \leq \nu$ (ν is* finer *than ν' or ν' is* coarser *than ν) if and only if $1_X : (X, \nu) \longrightarrow (X, \nu')$ is a contraction.*

Theorem 3.16 *Let $\varepsilon \in (0, \infty]$. Then the family of all ε-approach nearness on X forms a completely distributive complete lattice with respect to the partial order '$\leq$'. The zero of this lattice is the indiscrete ε-approach nearness ν_i on X and the unit is the discrete ε-approach nearness ν_d on X.*

Proof. Let $\{\nu_j : j \in J\}$ be a family of ε-approach nearness on X. Define $\nu_{\sup} : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ as follows: for $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$,

$$\nu_{\sup}(\mathcal{A}, \mathcal{B}) = \sup\{\inf_{i=1}^{n} \sup_{j \in J} \nu_j(\mathcal{A}_i, \mathcal{B}_k) : (\mathcal{A}_i)_{i=1}^n \in \mathfrak{C}(\mathcal{A}) \text{ and } (\mathcal{B}_k)_{k=1}^m \in \mathfrak{C}(\mathcal{B})\},$$

where $\mathfrak{C}(\mathcal{A})$ is the collection of all finite families $(\mathcal{A}_i)_{i=1}^n \subseteq \mathcal{P}^2(X)$ such that $\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n \prec \mathcal{A}$, and similarly $\mathfrak{C}(\mathcal{B})$ is defined. Then $\nu_{\sup}$ is an ε-approach nearness on X and is the supremum of the family of nearness $\{\nu_j : j \in J\}$ (techniques of the proof are similar to that of Theorem 3.8). Now we construct the infimum of the given family. Define $\nu_{\inf} : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ as follows: for $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$,

$$\nu_{\inf}(\mathcal{A}, \mathcal{B}) = \inf_{j \in J} \nu_j(cl(\mathcal{A}), cl(\mathcal{B})),$$

where $cl : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ is defined as: for $A \subseteq X$, $cl(A) = \bigcap\{B \subseteq X : A \subseteq B \text{ and } cl_{\nu_j}(B) = B, \text{ for all } j \in J\}$. Then $\nu_{\inf}$ is an ε-approach nearness on X and $cl_{\nu_{\inf}} = cl$. This $\nu_{\inf}$ is also the infimum of the given family $\{\nu_j : j \in J\}$ (proof follows similarly as in Proposition 3.10).

Corollary 3.17 *Let $\varepsilon \in (0, \infty]$. Then the family of all ε-approach merotopies on X forms a completely distributive complete lattice with respect to the partial order '$\leq$'. The zero of this lattice is the indiscrete ε-approach merotopy ν_i on X and the unit is the discrete ε-approach merotopy ν_d on X.*

Proof. If $\{\nu_j : j \in J\}$ is a family of ε-approach merotopies on X, then its supremum is defined in a similar manner as in the above theorem. The infimum of the given family $\nu_{\inf} : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \longrightarrow [0, \infty]$ is defined as follows: for $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$,

$$\nu_{\inf}(\mathcal{A}, \mathcal{B}) = \inf_{j \in J} \nu_j(\mathcal{A}, \mathcal{B}).$$

Concluding Remark. The present paper establishes a category theoretic foundation of the category $\boldsymbol{\varepsilon ANear}$ having objects ε-approach nearness spaces that are useful in the study of degree of nearness of nonempty sets.

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