Mathematica Aeterna, Vol. 3, 2013, no. 7, 579 - 591

Extinction properties of solutions for a p-Laplacian evolution equation with nonlinear source and strong absorption

Zhong Bo Fang*
fangzb7777@hotmail.com
School of Mathematical Sciences,
Ocean University of China,
Qingdao 266100, P.R. China

Mei Wang
wangmei600@126.com
School of Mathematical Sciences,
Ocean University of China,
Qingdao 266100, P.R. China

Gang Li
lgang502@163.com
School of Mathematical Sciences,
Ocean University of China,
Qingdao 266100, P.R. China

Abstract

This work is concerned with the extinction properties of solutions for a p-Laplacian evolution equation with source and strong absorption terms. We find the sufficient condition for the existence of extinction solutions and the corresponding decay estimate under suitable L^p-integral norm sense.

Mathematics Subject Classification: 35K65, 35K20, 35A70.
Keywords: p-Laplacian evolution equation, extinction, decay estimate.

1 Introduction

In this paper, we consider a class of p-Laplacian evolution equations

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u^q - \beta u^k, \qquad (x,t) \in \Omega \times (0,+\infty), \tag{1}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,+\infty), \tag{2}$$

$$u(x,0)=u_0(x), \qquad x\in\bar{\Omega}, \tag{3}$$

where $1<p<2$, $p-1\leq q<1$, $0<k<1$, $\lambda>0$, $\beta>0$, $\Omega\subset R^N(N\geq 1)$ is a bounded domain with smooth boundary, and initial data $u_0(x)\in L^\infty(\Omega)\bigcap W_0^{1,p}(\Omega)$ is a nonnegative function.

The problem (1) aries in the theory of quasiregular and quasiconformal mappings, stochastic control and non-Newtonian fluids, etc. In the non-Newtonian theory, the quantity p is a characteristic of the medium. Media with p > 2 are called dilatant fluids while p < 2 are called pseudoplastics. If p = 2, they are Newtonian fluids. Meanwhile, λu^q is called inner source term and $-\beta u^k(0<k<1)$ represents strong absorption term.

We are concerned only with the extinction solutions of problem (1)-(3). Extinction phenomenon is an important property for solutions of many evolutionary equations, especially for fast diffusion equations. In 1974, Kalashnikov [1] considered the Cauchy problem of a semilinear equation with absorption term $u_t=\Delta u-u^q$ and firstly introduced the definition of extinction for its solution, that is, there exists a finite time $T>0$ such that the solution is nontrivial on $(0,T)$ and then $u(x,t)\equiv 0$ for all $(x,t)\in\Omega\times[T,+\infty)$. In this case, T is called an extinction time. Later, many authors became interested in the extinction and nonextinction of all kinds of evolutionary equations. For the following parabolic equation without absorption term

$$u_t=\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\lambda u^q, \qquad (x,t)\in\Omega\times(0,+\infty),$$

where $\lambda\geq 0$ and $0<q\leq 1$. In case $\lambda=0$, Dibenedetto [2] and Yuan et al.[3] proved that the necessary and sufficient condition for the extinction to occur is $1<p<2$. For the case $\lambda>0$, Gu [4] proved that if $1<p<2$ or $0<q<1$, the solutions of the problem vanish in finite time, but if $p\geq 2$ and $q\geq 1$, there is nonextinction. Tian [5] and Yin et al.[6] showed that $q=p-1$ is the critical exponent of the weak solution. But all the results are limited to the local range and the higher dimensional space, while precise decay estimate has not been given.

Recently, Fang and Li [7] considered equation with linear absorption term

$$u_t=\operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)+\lambda|u|^{q-1}u-\beta u, \qquad (x,t)\in\Omega\times(0,+\infty),$$

where $1<p<2$, $0<m(p-1)\leq q<1$, $\lambda>0$, $\beta>0$. In the whole dimensional space, they showed that the extinction of the weak solution is determined by the competition of two nonlinear terms. They also obtained the exponential decay estimates which depend on the initial data, coefficients, and domains. Thereafter, they obtained the same results for a class of nonlocal problems, see [8,9]. The extinction and decay estimates for solutions to the p-Laplacian evolution equations with nonzero coefficients and strong absorption terms, like equation (1), are still being investigated.

Our main purpose is to establish the sufficient conditions about the extinction of solutions for the problem (1)-(3) in the whole dimensional space. By combining the L^p-integral norm estimate method and the technique of differential inequalities, we find that the extinction phenomena of solutions in our problem (1)-(3) is determined by the competition of nonlinear terms, and the decay estimates depend on the choices of initial data, coefficients and domain.

The rest of our paper is organized as follows. In Section 2, we give the preliminaries and main results for problem (1)-(3). Then the proofs are given in Section 3.

2 Preliminary Notes

Since equation (1) is singular when $1 < p < 2$, there is no classical solution in general. Hence, it is reasonable to find a weak solution of (1). To this end, we first give the following definitions of lower and upper nonnegative weak solutions of problem (1)-(3).

Definition 2.1 *We say that a non-negative nontrivial function $u(x,t)$ defined in $Q_T = \Omega \times (0,T)$ is a weak low(upper) solution of problem (1)-(3) if the following conditions hold:*
(i) $u \in C(0,T; L^\infty(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega)), u_t \in L^2(0,T; L^2(\Omega))$.
(ii) *For any* $0 < t < T$ *and any test function* $\varphi \in C_0^\infty(Q_T)$

$$\int_\Omega u(x,t)\varphi(x,t)dx \le (\ge) \int_\Omega u(x,0)\varphi(x,0)dx + \int_0^t \int_\Omega \{u\varphi_s - |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\}dxds$$
$$+ \int_0^t \int_\Omega \{\lambda u^q - \beta u^k\}\varphi(x,s)dxds,$$

(iii) $u(x,t) \le (\ge) u_0(x)$ *a.e.* $x \in \Omega$.
A function u is called a local weak solution of problem (1)-(3) if it is both a low solution and a upper solution for some $T > 0$.

Remark 2.1 *The existence and uniqueness of local nonnegative solution in time to problem (1)-(3) can be obtained by using the fixed point theorem or the standard parabolic regular theory to get a suitable estimate in the standard limiting process (see [2,10]). The proof is more or less standard, and so it is omitted here.*

Next, we recall two lemmas which are very important in the following proofs of our results. As for the proofs of these lemmas, we will not repeat them again (see [9,11]).

Lemma 2.2 *Let k and α be positive constants and $k < 1$. If $y(t)$ is a nonnegative absolutely continuous function on $t \in [0, \infty)$ satisfying the problem*

$$\frac{dy}{dt} + \alpha y^k \leq 0, \quad t \geq 0; \qquad y(0) \geq 0,$$

then we have the decay estimate

$$y(t) \leq [y^{1-k}(0) - \alpha(1-k)t]^{\frac{1}{1-k}}, \qquad t \in [0, T_*),$$

$$y(t) \equiv 0, \qquad t \in [T_*, +\infty),$$

where $T_* = \frac{y^{1-k}(0)}{\alpha(1-k)}$.

Lemma 2.3 *(Gagliardo-Nirenberg inequality) Suppose $u \in W_0^{k,m}(\Omega)$, $1 \leq m \leq \infty$ and $0 \leq j < k$, $1 \geq \frac{1}{r} \geq \frac{1}{m} - \frac{k}{N}$, then we have*

$$\|D^j u\|_q \leq C \|D^k u\|_m^\theta \|u\|_r^{1-\theta},$$

where C is a constant depending only on N, m, r, j, k, and q such that $\frac{1}{q} = \frac{j}{N} + \theta(\frac{1}{m} - \frac{k}{N}) + \frac{1-\theta}{r}$, $\theta \in [0, 1)$. While if $m < \frac{N}{k-j}$, then $q \in [\frac{Nr}{N+rj}, \frac{Nm}{N-(k-j)m}]$, if $m \geq \frac{N}{k-j}$, then $q \in [\frac{Nr}{N+rj}, +\infty]$.

3 Main Results

By using L^p-integral norm estimate method and the technique of differential inequality, we will obtain the sufficient conditions of extinction and the decay estimates for problem (1)-(3). Our detailed results are as follows:

Theorem 3.1 *Suppose that $0 < k < 1$, $0 < q = p - 1 < 1$, and λ_1 is the first eigenvalue of*

$$-div(|\nabla\varphi|^{p-2}\nabla\varphi) = \lambda\varphi^{p-1}, \ \ \varphi|_{\partial\Omega} = 0. \tag{4}$$

Then the weak solution of problem (1)-(3) vanishes in finite time for any nonnegative initial data provided that $\lambda < \lambda_1$ and β are sufficiently small.

Theorem 3.2 *If $0 < k < 1$, then the nonnegative weak solution of problem (1)-(3) vanishes in finite time provided that u_0 or λ is sufficiently small, and $q > \frac{(r+1)pk+N(p-1-k)}{(r+1)p+N(p-1-k)}$, where if $N = 1$ or 2, then $r = 1$ and if $N > 2$, then $r > p - 1$.*

Theorem 3.3 *Suppose that $0 < k < 1$ and $p - 1 > q \geq k$. Then the nonnegative nontrivial weak solution of problem (1)-(3) vanishes in finite time for any nonnegative initial data provided that β is sufficiently large.*

Remark 3.1 *Theorems 1-3 require that λ or β or u_0 should be sufficiently small (or large) and as for the decay estimates, we will give more concrete conditions which satisfy in the later proofs*

Remark 3.2 *One can see from Theorem 3.1-3.3 that the extinction of non-negative nontrivial weak solutions to problem (1)-(3) occurs when* $0 < k \leq q < 1$.

4 Proofs of the Main Results

In this section, we give detailed proofs of our main results to problem (1)-(3).
Proof of Theorem 3.1: We first consider the case $N = 1$ or 2. Multiplying (1) by u and integrating over Ω yield

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \int_\Omega |\nabla u|^p \, dx = \lambda \int_\Omega u^p \, dx - \beta\|u\|_{k+1}^{k+1}. \tag{5}$$

Since $\lambda_1 = inf_{0\neq v\in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p \, dx}{\int_\Omega |v|^p \, dx}$, we see that (5) becomes

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + (1 - \frac{\lambda}{\lambda_1})\|\nabla u\|_p^p + \beta\|u\|_{k+1}^{k+1} \leq 0. \tag{6}$$

By Lemma 2.3, we get the inequality

$$\|u\|_2 \leq C(N, p, k)\|u\|_{k+1}^{1-\theta_1}\|\nabla u\|_p^{\theta_1}, \tag{7}$$

where $\theta_1 = (\frac{1}{k+1} - \frac{1}{2})(\frac{1}{k+1} - \frac{1}{p} + \frac{1}{N})^{-1} = \frac{pN(1-k)}{2[p(k+1)+N(p-1-k)]}$. Since $N = 1$ or 2, $0 < k < 1$, and $1 < p < 2$, it can be easily seen that $0 < \theta_1 < 1$.

It then follows from (7) and Young's inequality that

$$\begin{aligned} \|u\|_2^{k_1} &\leq C(N, p, k)^{k_1}\|u\|_{k+1}^{k_1(1-\theta_1)}\|\nabla u\|_p^{k_1\theta_1} \\ &\leq C(N, p, k)^{k_1}(\eta_1\|\nabla u\|_p^p + C(\eta_1)\|\|u\|_{k+1}^{\frac{pk_1(1-\theta_1)}{p-k_1\theta_1}}), \end{aligned} \tag{8}$$

where $k_1 > 1$ and $\eta_1 > 0$ will be determined later.

If we choose $k_1 = \frac{p(k+1)}{p(1-\theta_1)+(k+1)\theta_1} = \frac{2p(k+1)+2N(p-k-1)}{2p+N(p-k-1)}$, then $1 < k_1 < 2$ and $\frac{pk_1(1-\theta_1)}{p-k_1\theta_1} = k + 1$. From (8) we have

$$\frac{\beta C(N, p, k)^{-k_1}}{C(\eta_1)}\|u\|_2^{k_1} \leq \frac{\eta_1\beta}{C(\eta_1)}\|\nabla u\|_p^p + \beta\|u\|_{k+1}^{k+1}. \tag{9}$$

By (6) and (9), we get the inequality

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + (1 - \frac{\lambda}{\lambda_1} - \frac{\eta_1\beta}{C(\eta_1)})\|\nabla u\|_p^p + \frac{\beta C(N, p, k)^{-k_1}}{C(\eta_1)}\|u\|_2^{k_1} \leq 0.$$

Here, we can choose η_1 and λ small enough so that

$$1-\frac{\lambda}{\lambda_1}-\frac{\eta_1\beta}{C(\eta_1)}\geq 0.$$

Setting $C_1=\frac{\beta C(N,p,k)^{-k_1}}{C(\eta_1)}$, we have $\frac{d}{dt}\|u\|_2+C_1\|u\|_2^{k_1-1}\leq 0$. By Lemma 1, we then obtain

$$\|u\|_2\leq[\|u_0\|_2^{2-k_1}-C_1(2-k_1)t]^{\frac{1}{2-k_1}}, t\in[0,T_1),$$

$$\|u\|_2=0,\quad t\in[T_1,+\infty),$$

where $T_1=\frac{\|u_0\|_2^{2-k_1}}{C_1(2-k_1)}$.

Secondly, we consider the case $N>2$. If $\frac{2N}{N+2}<p-1<1$, multiplying (1) by u^r(where $r>p-1$) and integrating over Ω yield

$$\frac{1}{r+1}\frac{d}{dt}\|u\|_{r+1}^{r+1}+\frac{rp^p}{(p+r-1)^p}\|\nabla u^{\frac{p+r-1}{p}}\|_p^p=\lambda\int_\Omega u^{p+r-1}\,dx-\beta\|u\|_{r+k}^{r+k}. \quad (10)$$

Using λ_1, (10) becomes

$$\frac{1}{r+1}\frac{d}{dt}\|u\|_{r+1}^{r+1}+(\frac{rp^p}{(p+r-1)^p}-\frac{\lambda}{\lambda_1})\|\nabla u^{\frac{p+r-1}{p}}\|_p^p+\beta\|u\|_{k+r}^{k+r}\leq 0. \quad (11)$$

By Lemma 2.3, we can also have

$$\|u\|_{r+1}\leq C(N,r,k)\|u\|_{k+r}^{1-\theta_2}\|\nabla u^{\frac{p+r-1}{p}}\|_p^{\frac{p\theta_2}{p+r-1}}, \quad (12)$$

where $\theta_2=\frac{p+r-1}{p}(\frac{1}{k+r}-\frac{1}{r+1})(\frac{1}{N}-\frac{1}{p}+\frac{p+r-1}{p}\cdot\frac{1}{k+r})^{-1}\frac{N(p+r-1)(1-k)}{(r+1)[p(k+r)+N(p-k-1)]}$. Since $0<k<1$, $\frac{2N}{N+2}<p-1<1$ and by the choice of r, it can be easily seen that $0<\theta_2<1$.

It then follows from (12) and Young's inequality that

$$\|u\|_{r+1}^{k_2}\leq C(N,r,k)^{k_2}\|u\|_{k+r}^{k_2(1-\theta_2)}\|\nabla u^{\frac{p+r-1}{p}}\|_p^{\frac{k_2p\theta_2}{p+r-1}}$$

$$\leq C(N,r,k)^{k_2}(\eta_2\|\nabla u^{\frac{p+r-1}{p}}\|_p^p+C(\eta_2)\|\|u\|_{k+1}^{\frac{k_2(1-\theta_2)(p+r-1)}{p+r-1-k_2\theta_2}}), \quad (13)$$

where $k_2>1$ and $\eta_2>0$ will be determined later.

If we choose $k_2=\frac{(p+r-1)(k+r)}{(p+r-1)(1-\theta_2)+(k+r)\theta_2}=\frac{(r+1)[p(k+r)+N(p-k-1)]}{p(r+1)+N(p-k-1)}$, there results $r<k_2<r+1$, $\frac{(p+r-1)k_2(1-\theta_2)}{p+r-1-k_2\theta_2}=k+r$. From (13) we have

$$\frac{\beta C(N,r,k)^{-k_2}}{C(\eta_2)}\|u\|_{r+1}^{k_2}\leq\frac{\eta_2\beta}{C(\eta_2)}\|\nabla u^{\frac{p+r-1}{p}}\|_p^p+\beta\|u\|_{k+r}^{k+r}. \quad (14)$$

By (14) and (11), we get

$$\frac{1}{r+1}\frac{d}{dt}\|u\|_{r+1}^{r+1}+(\frac{rp^p}{(p+r-1)^p}-\frac{\lambda}{\lambda_1}-\frac{\eta_2\beta}{C(\eta_2)})\|\nabla u^{\frac{p+r-1}{p}}\|_p^p+\frac{\beta C(N,r,k)^{-k_2}}{C(\eta_2)}\|u\|_{r+1}^{k_2}\leq 0.$$

Here, we can choose η_2 and λ small enough so that

$$(\frac{rp^p}{(p+r-1)^p}-\frac{\lambda}{\lambda_1}-\frac{\eta_2\beta}{C(\eta_2)})\geq 0.$$

Setting $C_2=\frac{\beta C(N,r,k)^{-k_2}}{C(\eta_2)}$, we have $\frac{d}{dt}\|u\|_{r+1}+C_2\|u\|_{r+1}^{k_2-r}\leq 0$. By Lemma 2.1, we then obtain

$$\|u\|_{r+1}\leq[\|u_0\|_{r+1}^{r+1-k_2}-C_2(r+1-k_2)t]^{\frac{1}{r+1-k_2}}, t\in[0,T_2),$$

$$\|u\|_{r+1}=0,\quad t\in[T_2,+\infty).$$

where $T_2=\frac{\|u_0\|_{r+1}^{r+1-k_2}}{C_2(r+1-k_2)}$.

If $0<p-1<\frac{N-2}{N+2}$, then multiply both sides of (1.1) by u^r $(r>\frac{N(2-p)}{2}-1)$ and integrate the result over Ω. By using the inequality above and a similar argument as above ,the following decay estimates can be obtained:

$$\|u\|_{r+1}\leq[\|u_0\|_{r+1}^{r+1-k_2}-\frac{C(N,k,r)^{-k_2}}{C(\eta_2)}(r+1-k_2)t]^{\frac{1}{r+1-k_2}}, t\in[0,T_2^*),$$

$$\|u\|_{r+1}=0, t\in[T_2^*,+\infty).$$

where $T_2^*=\frac{C(\eta_2)\|u_0\|_{r+1}^{r+1-k_2}}{C(N,k,s)^{-k_2}(r+1-k_2)}$.

Proof of Theorem 3.2: Assume that $q\leq 1$. If $N=1$ or 2, multiplying both sides of (1) by u and integrating the result over Ω yield the identity

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2+\int_\Omega|\nabla u|^p\,dx=\lambda\int_\Omega u^{q+1}\,dx-\beta\|u\|_{k+1}^{k+1}. \tag{15}$$

Then we substitute (9) into (15) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2+(1-\frac{\eta_1\beta}{c(\eta_1)})\|\nabla u\|_p^p+\frac{\beta C(N,p,k)^{-k_1}}{c(\eta_1)}\|u\|_2^{k_1}\leq\lambda\|u\|_{q+1}^{q+1},$$

The application of Hölder's inequality gives

$$\|u\|_{q+1}^{q+1}\leq|\Omega|^{\frac{1-q}{2}}\|u\|_2^{q+1}. \tag{16}$$

And we choose η_1 small enough such that $1-\frac{\eta_1\beta}{c(\eta_1)}\geq 0$, thus we get

$$\frac{d}{dt}\|u\|_2^2+[\frac{\beta C(N,p,k)^{-k_1}}{c(\eta_1)}-\lambda|\Omega|^{\frac{1-q}{2}}\|u\|_2^{q-k_1+1}]\|u\|_2^{k_1-1}\leq 0.$$

Therefore
$$\frac{d}{dt}\|u\|_2 + C_3\|u\|_2^{k_1-1} \le 0$$
provided that $\|u_0\|_2 \le (\frac{\beta C(N,p,k)^{-k_1}}{\lambda|\Omega|^{\frac{1-q}{2}}c(\eta_1)})^{\frac{1}{q+1-k_1}}$, and
$$q > k_1 - 1 = \frac{2pk + N(p-1-k)}{2p + N(p-1-k)},$$
where $C_3 = \frac{\beta C(N,p,k)^{-k_1}}{c(\eta_1)} - \lambda|\Omega|^{\frac{1-q}{2}}\|u_0\|_2^{q-k_1+1} > 0$.

By Lemma 2.2, we can obtain
$$\|u\|_2 \le [\|u_0\|_2^{2-k_1} - C_3(2-k_1)t]^{\frac{1}{2-k_1}}, t \in [0, T_3),$$
$$\|u\|_2 = 0, t \in [T_3, +\infty).$$
where $T_3 = \frac{\|u_0\|_2^{2-k_1}}{C_3(2-k_1)}$.

If $N > 2$ and $0 < p-1 < 1$, then multiplying (1) by u^r($r > p-1$) and integrating over Ω yield
$$\frac{1}{r+1}\frac{d}{dt}\|u\|_{r+1}^{r+1} + \frac{rp^p}{(p+r-1)^p}\|\nabla u^{\frac{p+r-1}{p}}\|_p^p = \lambda\int_\Omega u^{q+r}\,dx - \beta\|u\|_{r+k}^{r+k} \quad (17)$$
Substitute (14) into above equality to obtain
$$\frac{1}{r+1}\frac{d}{dt}\|u\|_{r+1}^{r+1} + (1 - \frac{\eta_2\beta}{c(\eta_2)})\|\nabla u^{\frac{p+r-1}{p}}\|_p^p + \frac{\beta C(N,r,k)^{-k_2}}{c(\eta_2)}\|u\|_{r+1}^{k_2} \le \lambda\|u\|_{q+r}^{q+r}.$$
The application of Hölder's inequality gives
$$\|u\|_{q+r}^{q+r} \le |\Omega|^{\frac{1-q}{r+1}}\|u\|_{r+1}^{q+r}. \quad (18)$$
And we choose η_2 small enough such that $1 - \frac{\eta_2\beta}{c(\eta_2)} \ge 0$, thus we get
$$\frac{d}{dt}\|u\|_{r+1}^{r+1} + [\frac{\beta C(N,r,k)^{-k_2}}{c(\eta_2)} - \lambda|\Omega|^{\frac{1-q}{r+1}}\|u\|_{r+1}^{q-k_2+r}]\|u\|_{r+1}^{k_2-r} \le 0,$$
Therefore
$$\frac{d}{dt}\|u\|_{r+1} + C_4\|u\|_{r+1}^{k_2-r} \le 0,$$
provided that $\|u_0\|_{r+1} \le (\frac{\beta C(N,r,k)^{-k_2}}{\lambda|\Omega|^{\frac{1-q}{1+r}}c(\eta_2)})^{\frac{1}{q+r-k_2}}$, and
$$q > k_2 - r = \frac{(r+1)pk + N(p-1-k)}{(r+1)p + N(p-1-k)},$$

where $C_4 = \frac{\beta C(N,r,k)^{-k_2}}{c(\eta_2)} - \lambda|\Omega|^{\frac{1-q}{r+1}}\|u_0\|_{r+1}^{q-k_2+r} > 0$.

By Lemma 2.2, we can obtain

$$\|u\|_{r+1} \le [\|u_0\|_{r+1}^{r+1-k_2} - C_4(r+1-k_2)t]^{\frac{1}{r+1-k_2}}, t \in [0, T_4),$$

$$\|u\|_{r+1} = 0, t \in [T_4, +\infty),$$

where $T_4 = \frac{\|u_0\|_{r+1}^{r+1-k_2}}{C_4(r+1-k_2)}$.

Since $r > p-1$, it follows that $2(r+1) > 2p$, and hence, if $k \ge p-1$, then $q > k_2 - r > p-1$.

Assume that $q > 1$. If λ_1 is the first eigenvalue of the boundary problem (4) and $\psi_1(x) \ge 0, \|\psi_1\|_\infty = 1$, is an eigenfunction corresponding to the eigenvalue λ_1, then for sufficiently small $a > 0$, it can be easily shown that $a\psi_1$ is an upper solution of problem (1)-(3) provided that $u_0(x) \le a\psi_1, x \in \Omega$. We then have $u(x,t) \le a\psi_1(x)$ for $t > 0$ by the comparison principle. Therefore, from equation (17), we can obtain the inequality

$$\frac{1}{r+1}\frac{d}{dt}\|u\|_{r+1}^{r+1}+(1-\lambda a^{q-p-1}C_0^{2|\Omega|^{1+\frac{2}{N}}}-\frac{\eta_2\beta}{c(\eta_2)})\|\nabla u^{\frac{p+r-1}{p}}\|_p^p+\frac{\beta C(N,r,k)^{-k_2}}{c(\eta_2)}\|u\|_{r+1}^{k_2} \le 0,$$

from which the following decay estimates can be obtained:

$$\|u\|_{r+1} \le [\|u_0\|_{r+1}^{r+1-k_2} - \frac{C(N,k,r)^{-k_2}}{C(\eta_2)}(r+1-k_2)t]^{\frac{1}{r+1-k_2}}, t \in [0, T_4^*),$$

$$\|u\|_{r+1} = 0, t \in [T_4^*, +\infty),$$

provided that

$$1 - \lambda a^{q-p-1}C_0^{2|\Omega|^{1+\frac{2}{N}}} - \frac{\eta_2\beta}{c(\eta_2)} \ge 0,$$

where $T_4^* = \frac{C(\eta_2)\|u_0\|_{r+1}^{r+1-k_2}}{C(N,k,s)^{-k_2}\beta(r+1-k_2)}$.

Remark 4.1 *Since the Sobolev embedding inequality cannot be used in the proof of The Theorem 2, it is not necessary to consider the cases where* $\frac{N-2}{N+2} \le p-1 < 1$ *and* $0 < p-1 < \frac{N-2}{N+2}$*, when* $N > 2$*. In addition, if* $k \ge p-1$*, the condition in Theorem 2 implies that* $q > p-1$*.*

Proof of Theorem 3.3: If $N = 1$ or 2, then multiplying both sides of (1) by u and integrating the result over Ω yield the equation

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \int_\Omega |\nabla u|^p\, dx = \lambda\int_\Omega u^{q+1}\, dx - \beta\|u\|_{k+1}^{k+1}. \tag{19}$$

By Lemma 2.3, we have the inequality

$$\|u\|_{q+1} \le C(N,k,q)\|u\|_{k+1}^{1-\theta_3}\|\nabla u\|_p^{\theta_3}, \tag{20}$$

where $\theta_3 = (\frac{1}{k+1} - \frac{1}{q+1})(\frac{1}{k+1} - \frac{1}{p} + \frac{1}{N})^{-1} = \frac{pN(q-k)}{(q+1)[p(k+1)+N(p-1-k)]} \in [0,1)$. Since $q < p-1$, it follows that $p-(q+1)\theta_3 > 0$. It then follows from (20) and Young's inequality that

$$\lambda\|u\|_{q+1}^{q+1} \le \lambda C^{q+1}(N,k,q)\|u\|_{k+1}^{(q+1)(1-\theta_3)}\|\nabla u\|_p^{(q+1)\theta_3}$$

$$\le \lambda C^{q+1}(N,k,q)(\eta_3\|\nabla u\|_p^p) + C(\eta_3\|u\|_{k+1}^{\frac{p(q+1)(1-\theta_3)}{p-(q+1)\theta_3}}), \tag{21}$$

where η_3 will be determined later. From (19) and (21), one can see that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + [1-\eta_3\lambda C(N,k,q)^{q+1}]\|\nabla u\|_p^p + \beta\|u\|_{k+1}^{k+1}$$

$$\le C(\eta_3)\lambda C(N,k,q)^{q+1}\|u\|_{k+1}^{\frac{p(q+1)(1-\theta_3)}{p-(q+1)\theta_3}}.$$

By Poincaré inequality we have $\|u\|_2^p \le \alpha\|u\|_p^p$, since $1 < p < 2$. Then we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + [1-\eta_3\lambda C(N,k,q)^{q+1}]\gamma^{-1}\alpha^{-1}\|u\|_2^p$$

$$+\|u\|_{k+1}^{k+1}[\beta \le C(\eta_3)\lambda C(N,k,q)^{q+1}\|u\|_{k+1}^{\alpha_1}],$$

where $\alpha_1 = \frac{p(q+1)(1-\theta_3)}{p-(q+1)\theta_3} - (k+1) \ge 0$.

We can choose η_3 small enough so that $C_5 = 1-\eta_3\lambda C(N,k,q)^{q+1} > 0$. Once η_3 is fixed, we may choose β large enough so that

$$\beta - C(\eta_3)\lambda C(N,k,q)^{q+1}\|u\|_{k+1}^{\alpha_1} \ge 0.$$

Hence, we have the inequality

$$\frac{d}{dt}\|u\|_2 + C_5\|u\|_2^{p-1} \le 0,$$

from which the following decay estimates can be obtained by a similar argument as the one used in the proof of Theorem 3.2:

$$\|u\|_2 \le [\|u_0\|_2^{2-p} - C_5(2-p)t]^{\frac{1}{2-p}},\ t \in [0,T_5),$$

$$\|u\|_2 = 0,\ t \in [T_5,+\infty),$$

where $T_5 = \frac{\|u_0\|_2^{2-p}}{C_5(2-p)}$.

Secondly, we consider the case $N > 2$. If $\frac{2N}{N+2} < p-1 < 1$, multiplying (1) by u^r(where $r > p-1$) and integrating over Ω yield

$$\frac{1}{r+1}\frac{d}{dt}\|u\|_{r+1}^{r+1} + \frac{rp^p}{(p+r-1)^p}\|\nabla u^{\frac{p+r-1}{p}}\|_p^p = \lambda\int_\Omega u^{q+r}\,dx - \beta\|u\|_{r+k}^{r+k}. \tag{22}$$

By Lemma 2.3, we can also have

$$\|u\|_{q+r} \leq C(N,q,r,k)\|u\|_{k+r}^{1-\theta_4}\|\nabla u^{\frac{p+r-1}{p}}\|_p^{\frac{p\theta_4}{p+r-1}}, \tag{23}$$

where $\theta_4 = \frac{p+r-1}{p}(\frac{1}{k+r} - \frac{1}{r+q})(\frac{1}{N} - \frac{1}{p} + \frac{p+r-1}{p} \cdot \frac{1}{k+r})^{-1} = \frac{N(p+r-1)(q-k)}{(r+q)[p(k+r)+N(p-k-1)]} \in [0,1)$. Since $q < p-1$, we have $p-1+r-(q+r)\theta_4 > 0$. It then follows from (23) and Young's inequality that

$$\lambda\|u\|_{q+r}^{q+r} \leq \lambda C(N,q,r,k)^{q+r}\|u\|_{k+r}^{(q+r)(1-\theta_4)}\|\nabla u^{\frac{p+r-1}{p}}\|_p^{\frac{p(q+r)\theta_4}{p+r-1}}$$

$$\leq \lambda C(N,q,r,k)^{q+r}(\eta_4\|\nabla u^{\frac{p+r-1}{p}}\|_p^p + C(\eta_4)\|u\|_{k+r}^{\frac{(q+r)k_2(1-\theta_4)(p+r-1)}{p+r-1-(q+r)\theta_4}}), \tag{24}$$

where η_4 will be determined later. Using Sobolev embedding theorem yields

$$\|u^{\frac{p+r-1}{p}}\|_s \leq \gamma\|\nabla u^{\frac{p+r-1}{p}}\|_p,$$

which leads to

$$\gamma^{-p}\|u\|_{\frac{(p+r-1)s}{p}}^{p+r-1} \leq \|\nabla u^{\frac{p+r-1}{p}}\|_p^p.$$

Choosing $s = \frac{p(r+1)}{p+r-1}$ gives

$$\gamma^{-p}\|u\|_{r+1}^{p+r-1} \leq \|\nabla u^{\frac{p+r-1}{p}}\|_p^p. \tag{25}$$

From (22), (24) and (25), one can see that

$$\frac{1}{r+1}\frac{d}{dt}\|u\|_{r+1}^{r+1} + [\frac{rp^p}{(p+r-1)^p} - \lambda C(N,q,r,k)^{q+r}\eta_4]\gamma^{-p}\|u\|_{r+1}^{p+r-1}$$

$$+\|u\|_{r+k}^{r+k}[\beta - C(\eta_4)\lambda C(N,k,q,r)^{q+r}\|u\|_{k+r}^{\alpha_2}] \leq 0,$$

where $\alpha_2 = \frac{(q+r)k_2(1-\theta_4)(p+r-1)}{p+r-1-(q+r)\theta_4} - (k+r) \geq 0$. We can choose η_4 small enough so that $C_6 = [\frac{rp^p}{(p+r-1)^p} - \lambda C(N,q,r,k)^{q+r}\eta_4]\gamma^{-p} > 0$. Once η_4 is fixed, we can choose β large enough so that

$$\beta - C(\eta_4)\lambda C(N,k,q,r)^{q+r}\|u\|_{k+r}^{\alpha_2} \geq 0.$$

Hence, we can obtain the inequality

$$\frac{d}{dt}\|u\|_{r+1} + C_6\|u\|_{r+1}^{p-1} \leq 0,$$

from which the following decay estimates can be obtained:

$$\|u\|_{r+1} \leq [\|u_0\|_{r+1}^{2-p} - C_5(2-p)t]^{\frac{1}{2-p}},\ t \in [0,T_6),$$

$$\|u\|_{r+1} = 0,\ t \in [T_6, +\infty),$$

where $T_6 = \frac{\|u_0\|_{r+1}^{2-p}}{C_6(2-p)}$.

Similarly, one can obtain the following decay estimates for $0 < p-1 < \frac{N-2}{N+2}$:

$$\|u\|_{r+1} \leq [\|u_0\|_{r+1}^{2-p} - [\frac{rp^p}{(p+r-1)^p} - \lambda C(N,q,r,k)^{q+r}\eta_4 C_{00}^{-p}](2-p)t]^{\frac{1}{2-p}},\ t \in [0, T_6^*),$$

$$\|u\|_{r+1} = 0,\ t \in [T_6^*, +\infty),$$

where $T_6^* = \frac{\|u_0\|_{r+1}^{2-p}}{[\frac{rp^p}{(p+r-1)^p} - \lambda C(N,q,r,k)^{q+r}\eta_4 C_{00}^{-p}](2-p)}$.

Remark 4.2 *Theorems 3.1-3.3 all require* λ, u_0 *to be sufficiently small or* β *to be sufficiently large.*

Acknowledgments

This work is supported by the Natural Science Foundation of Shandong Province of China (ZR2012AM018) and the Fundamental Research Funds for the Central Universities (No.201362032). The authors would like to deeply thank all the reviewers for their insightful and constructive comments.

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Received: August, 2013