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A Generalization of Some Integral Inequalities Similar to Hardy's Inequality

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Abstract

In 2012, Sulaiman gave some integral inequalities similar to Hardy's inequality. In this paper, we present a generalization of the inequalities.

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1 Introduction

In 1920, Hardy [1] posed the inequality as follows.

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx$$

where $f \geq 0$, $p > 1$, and $F(x) = \displaystyle\int_a^x f(t)dt$.

In 2012, Sulaiman [2] presented the inequalities similar to above inequality as follows.

$$p\int_a^b \left(\frac{F(x)}{x}\right)^p dx \leq (b-a)^p \int_a^b \left(\frac{f(x)}{x}\right)^p dx - \int_a^b \left(1-\frac{a}{x}\right)^p f^p(x)dx \quad (1)$$

where $f > 0$ on $[a,b] \subseteq (0,\infty)$, $p \geq 1$, and $F(x) = \displaystyle\int_a^x f(t)dt$.

$$p\int_a^b \left(\frac{F(x)}{x}\right)^p dx \geq \left(1-\frac{a}{b}\right)^p \int_a^b f^p(x)dx - \frac{1}{b^p}\int_a^b (x-a)^p f^p(x)dx \quad (2)$$

where $f > 0$ on $[a, b] \subseteq (0, \infty)$, $0 < p < 1$, and $F(x) = \int_a^x f(t)dt$.

In this paper, we present a generalization of both inequality (1) and inequality (2).

2 Main Results

Theorem 2.1. *Assume that $f > 0$ on $[a, b] \subseteq (0, \infty)$, and $p \geq 1$, $q > 0$. Define*

$$F(x) = \int_a^x f(t)dt.$$

Then

$$p \int_a^b \frac{F^p(x)}{x^q} dx \leq (b-a)^p \int_a^b \frac{f^p(x)}{x^q} dx - \int_a^b \frac{(x-a)^p}{x^q} f^p(x) dx.$$

Proof. Note that $\frac{1}{p} + \frac{p-1}{p} = 1$. By the assumption and the Hölder's inequality, we have

$$\begin{aligned}
\int_a^b \frac{F^p(x)}{x^q} dx &= \int_a^b x^{-q} \left(\int_a^x f(t)dt \right)^p dx \\
&\leq \int_a^b x^{-q} \left(\left(\int_a^x f^p(t)dt \right)^{\frac{1}{p}} \left(\int_a^x dt \right)^{\frac{p-1}{p}} \right)^p dx \\
&= \int_a^b x^{-q} \left(\left(\int_a^x f^p(t)dt \right)^{\frac{1}{p}} (x-a)^{\frac{p-1}{p}} \right)^p dx \\
&= \int_a^b x^{-q} \left(\int_a^x f^p(t)dt \right) (x-a)^{p-1} dx \\
&= \int_a^b \int_a^x x^{-q} (x-a)^{p-1} f^p(t) dt dx \\
&= \int_a^b \int_t^b x^{-q} (x-a)^{p-1} f^p(t) dx dt.
\end{aligned}$$

And then

$$\begin{aligned}\int_a^b \frac{F^p(x)}{x^q}dx &\le \int_a^b \int_t^b t^{-q}(x-a)^{p-1} f^p(t)dxdt \\ &= \int_a^b t^{-q} f^p(t)\left(\int_t^b (x-a)^{p-1}dx\right)dt \\ &= \int_a^b t^{-q} f^p(t)\left(\frac{(b-a)^p-(t-a)^p}{p}\right)dt \\ &= \frac{1}{p}\left((b-a)^p \int_a^b \frac{f^p(t)}{t^q}dt - \int_a^b \frac{(t-a)^p}{t^q} f^p(t)dt\right).\end{aligned}$$

□

Theorem 2.2. *Assume that* $f > 0$ *on* $[a,b] \subseteq (0,\infty)$*, and* $0 < p < 1$*,* $q > 0$*. Define*

$$F(x) = \int_a^x f(t)dt.$$

Then

$$p\int_a^b \frac{F^p(x)}{x^q}dx \ge \frac{(b-a)^p}{b^q}\int_a^b f^p(x)dx - \frac{1}{b^q}\int_a^b (x-a)^p f^p(x)dx.$$

Proof. Note that $\frac{1}{p} + \frac{p-1}{p} = 1$. By the assumption and the reverse Hölder's inequality, we have

$$\begin{aligned}\int_a^b \frac{F^p(x)}{x^q}dx &= \int_a^b x^{-q}\left(\int_a^x f(t)dt\right)^p dx \\ &\ge \int_a^b x^{-q}\left(\left(\int_a^x f^p(t)dt\right)^{\frac{1}{p}}\left(\int_a^x dt\right)^{\frac{p-1}{p}}\right)^p dx \\ &= \int_a^b x^{-q}\left(\left(\int_a^x f^p(t)dt\right)^{\frac{1}{p}}(x-a)^{\frac{p-1}{p}}\right)^p dx \\ &= \int_a^b x^{-q}\left(\int_a^x f^p(t)dt\right)(x-a)^{p-1}dx \\ &= \int_a^b \int_a^x x^{-q}(x-a)^{p-1} f^p(t)dtdx \\ &= \int_a^b \int_t^b x^{-q}(x-a)^{p-1} f^p(t)dxdt.\end{aligned}$$

And then

$$\begin{aligned}\int_a^b \frac{F^p(x)}{x^q} dx &\geq \int_a^b \int_t^b b^{-q} (x-a)^{p-1} f^p(t) dx dt \\ &= b^{-q} \int_a^b f^p(t) \left(\int_t^b (x-a)^{p-1} dx \right) dt \\ &= b^{-q} \int_a^b f^p(t) \left(\frac{(b-a)^p - (t-a)^p}{p} \right) dt \\ &= \frac{b^{-q}}{p} \left((b-a)^p \int_a^b f^p(t) dt - \int_a^b (t-a)^p f^p(t) dt \right).\end{aligned}$$

□

Corollary 2.3. [2] *Assume that $f > 0$ on $[a,b] \subseteq (0,\infty)$. Define*

$$F(x) = \int_a^x f(t) dt.$$

It follows that

(i) if $p \geq 1$ then

$$p \int_a^b \left(\frac{F(x)}{x} \right)^p dx \leq (b-a)^p \int_a^b \left(\frac{f(x)}{x} \right)^p dx - \int_a^b \left(1 - \frac{a}{x} \right)^p f^p(x) dx,$$

and

(ii) if $0 < p < 1$, then

$$p \int_a^b \left(\frac{F(x)}{x} \right)^p dx \geq \left(1 - \frac{a}{b} \right)^p \int_a^b f^p(x) dx - \frac{1}{b^p} \int_a^b (x-a)^p f^p(x) dx.$$

Proof. This follows from Theorem 2.1 and 2.2 where $q = p$. □

References

[1] G. H. Hardy, Note on a Theorem of Hilbert, Math. Z., 1920, **6**, 314–317.

[2] W. T. Sulaiman, Reverses of Minkowski's, Hölder's, and Hardy's integral inequalities, Int. J. Mod. Math. Sci., 2012, **1**(1), 14–24.

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