Mathematica Aeterna, Vol. 3, 2013, no. 8, 617 - 624

Harmonic Mappings for which Second Dilatation is Janowski Functions

Emel Yavuz Duman

İstanbul Kültür University
Department of Mathematics and Computer Science
Ataköy Campus
34156, İstanbul, Turkey

Yaşar Polatoğlu

İstanbul Kültür University
Department of Mathematics and Computer Science
Ataköy Campus
34156, İstanbul, Turkey

Yasemin Kahramaner

İstanbul Commerce University
Department of Mathematics
Üsküdar Campus
34672, İstanbul, Turkey

Maslina Darus

Universiti Kebangsaan Malaysia
School of Mathematical Sceinces
43600 Bangi, Selangor D. Ehsan
Kuala Lumpur, Malaysia

Abstract

In the present paper we extend the fundamental property that if $h(z)$ and $g(z)$ are regular functions in the open unit disc $\mathbb{D}$ with the properties $h(0) = g(0) = 0$, h maps $\mathbb{D}$ onto many-sheeted region which is starlike with respect to the origin, and $\text{Re}\frac{g'(z)}{h'(z)} > 0$, then $\text{Re}\frac{g(z)}{h(z)} > 0$, introduced by R.J. Libera [5] to the Janowski functions and give some applications of this to the harmonic functions.

Mathematics Subject Classification: 30C45; 30C55

Keywords: Harmonic mappings, Janowski functions, growth theorem, distortion theorem, coefficient inequality

1 Introduction

Let Ω be the class of functions $\phi(z)$ regular in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

For arbitrary fixed numbers A, B, $-1 < A \leq 1$, $-1 \leq B < A$ we denote by $\mathcal{P}(A, B)$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ regular in $\mathbb{D}$ such that $p(z)$ is in $\mathcal{P}(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)} \tag{1}$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$.

Let $\mathcal{S}^*(A, B)$ denote the family of functions $s(z) = z + c_2 z^2 + \cdots$ regular in $\mathbb{D}$ such that $s(z)$ in $\mathcal{S}^*(A, B)$ if and only if

$$z\frac{s'(z)}{s(z)} = p(z) \tag{2}$$

for some $p(z)$ in $\mathcal{P}(A, B)$ and all $z \in \mathbb{D}$. We note that every function in this family maps the unit disc univalently onto a region wihich is starlike with respect to the origin.

Let $s_1(z) = z + d_2 z^2 + \cdots$ and $s_2(z) = z + e_2 z^2 + \cdots$ be analytic functions in $\mathbb{D}$. If there exists $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$, then we say that $s_1(z)$ is subordinate to $s_2(z)$ and write $s_1(z) \prec s_2(z)$ so that $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$.

Finally, univalent harmonic functions are generalization of univalent analytic functions the point of the departure is the canonical representation

$$f = h(z) + \overline{g(z)}, \ g(0) = 0 \tag{3}$$

of a harmonic function f in the unit disc $\mathbb{D}$ as the sum of an analytic function $h(z)$ and conjugate of an analytic function $g(z)$. With the convention that $g(0) = 0$, the representation is unique. The power series expansions of $h(z)$ and $g(z)$ are denoted by

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \ g(z) = \sum_{n=0}^{\infty} b_n z^n. \tag{4}$$

If f is sense-preserving harmonic mapping of $\mathbb{D}$ onto some other region, then by Lewy's theorem its Jacobian is strictly positive, i.e,

$$\mathcal{J}_{f(z)} = |h'(z)|^2 - |g'(z)|^2 > 0.$$

Equivalently, the inequality $|g'(z)| < |h'(z)|$ holds for all $z \in \mathbb{D}$. This shows, in particular, that $h'(0) \neq 0$ and $h'(0) = 1$. The class of all sense-preserving harmonic mapping of the disc with $a_0 = b_0 = 0$, $a_1 = 1$ will be denoted by $\mathcal{S}_{\mathcal{H}}$. Thus $\mathcal{S}_{\mathcal{H}}$ contains the standard class $\mathcal{S}$ of analytic univalent functions. Although the analytic part $h(z)$ of a function $f \in \mathcal{S}_{\mathcal{H}}$ is locally univalent, it will be come apparent that it not be univalent. The class of functions $f \in \mathcal{S}_{\mathcal{H}}$ with $g'(0) = 0$ will be denoted by $\mathcal{S}^0_{\mathcal{H}}$. At the same time, we note that $\mathcal{S}_{\mathcal{H}}$ is a normal family and $\mathcal{S}^0_{\mathcal{H}}$ is a compact normal family. For details, see [2].

Now, we consider the following class of harmonic mappings in the plane

$$\mathcal{S}^*_{\mathcal{HPST}} = \Big\{ f = h(z) + \overline{g(z)} \Big| f \in \mathcal{S}_{\mathcal{H}},\ h(z) \in S^*(A,B),$$
$$w(z) = \frac{g'(z)}{b_1 h'(z)} \in \mathcal{P}(A,B), -1 \leq B < A \leq 1 \Big\}.$$

In this paper we will investigate the subclass $\mathcal{S}^*_{\mathcal{HPST}}$. We will need the following theorems in the sequel.

Theorem 1.1 *[4] Let $h(z)$ be an element of $\mathcal{S}^*(A,B)$, then*

$$C(r,-A,-B) \leq |h(z)| \leq C(r,A,B), \tag{5}$$

where

$$C(r,A,B) = \begin{cases} r(1+Br)^{\frac{A-B}{B}}, & B \neq 0, \\ re^{Ar}, & B = 0. \end{cases}$$

These bounds are sharp, being attained at the point $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, by

$$f_* = z f_0(z; -A, -B), \tag{6}$$

$$f^* = z f_0(z; A, B), \tag{7}$$

respectively, where

$$f_0(z;A,B) = \begin{cases} (1+Be^{-i\theta}z)^{\frac{A-B}{B}}, & B \neq 0, \\ re^{Ae^{-i\theta}z}, & B = 0. \end{cases}$$

Theorem 1.2 *[6] If $h(z) = z + a_2 z^2 + \cdots$ belongs to $\mathcal{S}^*(A,B)$, then*

$$|a_n| \leq \begin{cases} \prod_{k=0}^{n-2} \frac{|(A-B)+kB|}{k+1}, & B \neq 0, \\ \prod_{k=0}^{n-2} \frac{|A|}{k+1}, & B \neq 0. \end{cases}$$

These bound are sharp because the extremal function is

$$f_*(z) = \begin{cases} (1+Bz)^{\frac{A-B}{B}}, & B \neq 0, \\ re^{Az}, & B = 0. \end{cases}$$

Lemma 1.3 (Jack's Lemma, [3]) *Let $\phi(z)$ be regular in the unit disc $\mathbb{D}$, with $\phi(0) = 0$. Then if $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_1, one has $z_1\phi'(z_1) = k\phi(z_1)$ for some $k \geq 1$.*

2 Main Results

Theorem 2.1 *If $h(z)$ and $g(z)$ are regular in $\mathbb{D}$ such that $h(0) = g(0) = 0$. If $h(z) \in \mathcal{S}^*(A, B)$ and $\frac{g'(z)}{b_1 h'(z)} \in \mathcal{P}(A, B)$, then $\frac{g(z)}{b_1 h(z)} \in \mathcal{P}(A, B)$.*

Proof. Since the linear transformation $\frac{1+Az}{1+Bz}$ maps $|z| = r$ onto the disc with the center $C(r) = \left(\frac{1-ABr^2}{1-B^2r^2}, 0\right)$ and the radius $\rho(r) = \frac{(A-B)r}{1-B^2r^2}$ and using the subordination principle we can write

$$\frac{g'(z)}{h'(z)} \prec b_1 \frac{1+Az}{1+Bz}$$

so

$$\frac{1}{b_1}\frac{g'(z)}{h'(z)} \prec \frac{1+Az}{1+Bz} \Rightarrow \left|\frac{1}{b_1}\frac{g'(z)}{h'(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}$$

thus

$$\left|\frac{g'(z)}{h'(z)} - \frac{b_1(1-ABr^2)}{1-B^2r^2}\right| \leq \frac{|b_1|(A-B)r}{1-B^2r^2}. \tag{8}$$

The inequality (8) shows that the values of $g'(z)/h'(z)$ are in the disc

$$\mathbb{D}_r(b_1) = \begin{cases} \left\{ \frac{g'(z)}{h'(z)} \middle| \left|\frac{g'(z)}{h'(z)} - \frac{b_1(1-ABr^2)}{1-B^2r^2}\right| \leq \frac{|b_1|(A-B)r}{1-B^2r^2} \right\}, & B \neq 0, \\ \left\{ \frac{g'(z)}{h'(z)} \middle| \left|\frac{g'(z)}{h'(z)} - b_1\right| \leq |b_1|Ar \right\}, & B = 0. \end{cases}$$

Now we define a function $\phi(z)$ by

$$\frac{g(z)}{h(z)} = b_1 \frac{1+A\phi(z)}{1+B\phi(z)}.$$

Then $\phi(z)$ is analytic in $\mathbb{D}$, $\phi(0) = 0$. On the other hand since $h(z) \in \mathcal{S}^*(A, B)$ then

$$\mathbb{D}_r = \begin{cases} \left|z\frac{h'(z)}{h(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}, & B \neq 0, \\ \left|z\frac{h'(z)}{h(z)} - 1\right| \leq Ar, & B = 0 \end{cases}$$

for all $|z| = r < 1$. Thus, for a point z_1 on the bound of this disc we have

$$z_1\frac{h'(z_1)}{h(z_1)} - \frac{1-ABr^2}{1-B^2r^2} = \frac{(A-B)r}{1-B^2r^2}e^{i\theta}, \quad B \neq 0,$$
$$z_1\frac{h'(z_1)}{h(z_1)} - 1 = Are^{i\theta}, \quad B \neq 0,$$

or

$$\frac{h(z_1)}{z_1 h(z_1)} = \frac{1-B^2r^2}{[1-ABr^2] + (A-B)re^{i\theta}} \in \partial\mathbb{D}_r, \quad B \neq 0,$$
$$\frac{h(z_1)}{z_1 h(z_1)} = \frac{1}{1+Are^{i\theta}} \in \partial\mathbb{D}_r, \quad B \neq 0,$$

where $\partial\mathbb{D}_r$ is the boundary of the disc $\mathbb{D}_r$. Therefore, by Jack's lemma, $z_1\phi'(z_1) = k\phi(z_1)$ and $k \geq 1$, we have that

$$w(z_1) = \frac{g'(z_1)}{b_1 h'(z_1)} = \begin{cases} \frac{1+A\phi(z_1)}{1+B\phi(z_1)} + \frac{(A-B)k\phi(z_1)}{(1+B\phi(z_1))^2}\frac{1-B^2r^2}{(1-ABr^2)+(A-B)re^{i\theta}} \notin w(\mathbb{D}_r(b_1)), & B \neq 0, \\ 1 + A\phi(z_1) + Ak\phi(z_1)\frac{1}{1+Are^{i\theta}} \notin w(\mathbb{D}_r(b_1)), & B = 0, \end{cases} \tag{9}$$

because $|\phi(z_1)| = 1$ and $k \geq 1$. But this is a contradiction to the condition $\frac{g'(z)}{h'(z)} \prec b_1\frac{1+Az}{1+Bz}$ and so we have $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Lemma 2.2 *Let $f = h(z) + \overline{g(z)} \in \mathcal{S}_{\mathcal{H}}$, then for a function defined by $\omega(z) = \frac{g'(z)}{h'(z)}$ we have*

$$\frac{|b_1| - r}{1 - |b_1|r} \leq |\omega(z)| \leq \frac{|b_1| + r}{1 + |b_1|r}, \tag{10}$$

$$\frac{(1-r^2)(1-|b_1|^2)}{(1+|b_1|r)^2} \leq 1 - |\omega(z)|^2 \leq \frac{(1-r^2)(1-|b_1|^2)}{(1-|b_1|r)^2}, \tag{11}$$

and

$$\frac{(1-r)(1+|b_1|)}{1-|b_1|r} \leq 1 + |\omega(z)| \leq \frac{(1+r)(1+|b_1|)}{1+|b_1|r} \tag{12}$$

for all $|z| = r < 1$.

Proof. Since $f = h(z) + \overline{g(z)} \in \mathcal{S}_{\mathcal{H}}$, it follows that

$$\omega(z) = \frac{g'(z)}{h'(z)} = \frac{b_1 + 2b_2 z + \cdots}{1 + 2a_2 z + \cdots} \Rightarrow \omega(0) = b_1 \Rightarrow |\omega(z)| < 1 \Rightarrow |\omega(0)| = |b_1| < 1.$$

So the function

$$\phi(z) = \frac{\omega(z) - \omega(0)}{1 - \overline{\omega(0)}\omega(z)} = \frac{\omega(z) - b_1}{1 - \overline{b_1}\omega(z)}$$

satisfies the conditions of Schwarz lemma. Therefore we have

$$\omega(z) = \frac{b_1 + \phi(z)}{1 + \overline{b_1}\phi(z)} \text{ if and only if } \omega(z) \prec \frac{b_1 + z}{1 + \overline{b_1}z} \ (z \in \mathbb{D}). \tag{13}$$

On the other hand, the linear transformation $\left(\frac{b_1+z}{1+\overline{b_1}z}\right)$ maps $|z| = r$ onto the disc with the center $C(r) = \left(\frac{(1-r^2)\text{Re}(b_1)}{1-|b_1|^2r^2}, \frac{(1-r^2)\text{Im}(b_1)}{1-|b_1|^2r^2}\right)$ with the radius $\rho(r) = \frac{(1-|b_1|^2)r}{1-|b_1|^2r^2}$. Then, we have

$$\left|\omega(z) - \frac{b_1(1-r^2)}{1-|b_1|^2}\right| \leq \frac{(1-|b_1|^2)r}{1-|b_1|^2r^2}, \tag{14}$$

which gives (10), (11) and (13).

Corollary 2.3 *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}^*_{\mathcal{HPST}}$, then*

$$\begin{array}{rcll} |b_1|(1-Ar)^2(1-Br)^{\frac{A-3B}{B}} & \leq |g'(z)| \leq & |b_1|(1+Ar)^2(1+Br)^{\frac{A-3B}{B}}, & B \neq 0, \\ |b_1|(1-Ar)^2 e^{-Ar} & \leq |g'(z)| \leq & |b_1|(1+Ar)^2 e^{Ar}, & B = 0, \end{array} \quad (15)$$

and

$$\begin{array}{rcll} |b_1| r(1-Ar)(1-Br)^{\frac{A-2B}{B}} & \leq |g(z)| \leq & |b_1| r(1+Ar)(1+Br)^{\frac{A-2B}{B}}, & B \neq 0, \\ |b_1|(1-Ar)e^{-Ar} & \leq |g(z)| \leq & |b_1|(1+Ar)e^{Ar}, & B = 0, \end{array} \quad (16)$$

for all $|z| = r < 1$.

Proof. Since $h(z) \in \mathcal{S}^*(A,B)$, then we have

$$\begin{array}{rll} \left| z\frac{h'(z)}{h(z)} - \frac{1-ABr^2}{1-B^2r^2} \right| & \leq \frac{(A-B)r}{1-B^2r^2} \;\Rightarrow\; \frac{1-Ar}{1-Br} \leq \left| z\frac{h'(z)}{h(z)} \right| \leq \frac{1+Ar}{1+Br}, & B \neq 0, \\ \left| z\frac{h'(z)}{h(z)} - 1 \right| & \leq Ar \;\Rightarrow 1 - Ar \leq \left| z\frac{h'(z)}{h(z)} \right| \leq 1 + Ar, & B = 0, \end{array} \quad (17)$$

for all $z \in \mathbb{D}$. Using Theorem 1.1 and after simple calculations we get

$$\begin{array}{rcll} (1-Ar)(1-Br)^{\frac{A-2B}{B}} & \leq |h'(z)| \leq & (1+Ar)(1+Br)^{\frac{A-2B}{B}}, & B \neq 0, \\ (1-Ar)e^{-Ar} & \leq |h'(z)| \leq & (1+Ar)e^{Ar}, & B = 0. \end{array} \quad (18)$$

On the other hand, if we use Theorem 2.1, then we can write

$$\begin{array}{rcll} F(r,-A,-B,|b_1|) & \leq \left| \frac{g(z)}{h(z)} \right| \leq & F(r,A,B,|b_1|), & B \neq 0, \\ F(r,-A,|b_1|) & \leq \left| \frac{g(z)}{h(z)} \right| \leq & F(r,A,|b_1|), & B = 0, \end{array} \quad (19)$$

and

$$\begin{array}{rcll} F(r,-A,-B) & \leq \left| \frac{g'(z)}{h'(z)} \right| \leq & F(r,A,B), & B \neq 0, \\ F(r,-A) & \leq \left| \frac{g'(z)}{h'(z)} \right| \leq & F(r,A), & B = 0, \end{array} \quad (20)$$

for all $|z| = r < 1$, where $F(r,A,B,|b_1|) = \frac{|b_1|(1+Ar)}{1+Br}$ and $F_1(r,A,|b_1|) = |b_1|(1+Ar)$. Considering Theorem 1.1, and equations (18), (19) and (20), we get (15) and (16).

Corollary 2.4 *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}^*_{\mathcal{HPST}}(A,B)$, then*

$$\begin{array}{l} |b_1|(1-Ar)^2(1-Br)^{\frac{2(A-2B)}{B}} \dfrac{(1-r^2)(1-|b_1|^2)}{(1+|b_1|r)^2} \leq \mathcal{J}_{f(z)} \\ \qquad \leq |b_1|(1+Ar)^2(1-Br)^{\frac{2(A-2B)}{B}} \dfrac{(1-r^2)(1-|b_1|^2)}{(1-|b_1|r)^2}, B \neq 0, \\ |b_1|(1-Ar)^2 e^{-2Ar} \dfrac{(1-r^2)(1-|b_1|^2)}{(1+|b_1|r)^2} \leq \mathcal{J}_{f(z)} \\ \qquad \leq |b_1|(1+Ar)^2 e^{2Ar} \dfrac{(1-r^2)(1-|b_1|^2)}{(1-|b_1|r)^2}, B = 0, \end{array} \quad (21)$$

for all $|z| = r < 1$*, where*

Proof. This is a consequence of Lemma 2.2 and the inequalities in (18).

Corollary 2.5 *Let* $f = h(z) + \overline{g(z)}$ *be an element of* $\mathcal{S}^*_{\mathcal{HPST}}(A, B)$*, then*

$$\begin{aligned}
&|b_1|\frac{\left(-\frac{|b_1|}{B}\right)^{1-\frac{A}{B}}{}_2F_1\left[2-\frac{A}{B}, 2-\frac{A}{B}, 3-\frac{A}{B}, \frac{B+|b_1|}{B}\right]}{A-2B}\\
&-|b_1|\frac{1}{2}(1+A)r^2F_1\left[2, 2-\frac{A}{B}, 1, 3, Br, -|b_1|r\right]\\
&+|b_1|\frac{1}{3}Ar^3F_1\left[3, 2-\frac{A}{B}, 1, 4, Br, -|b_1|r\right]\\
&+|b_1|\frac{|b_1|(1-Br)^{\frac{A}{B}}\left(\frac{|b_1|(-1+Br)}{B(1+|b_1|r)}\right)^{-\frac{A}{B}}{}_2F_1\left[2-\frac{A}{B}, 2-\frac{A}{B}, 3-\frac{A}{B}, \frac{B+|b_1|}{B(1+|b_1|r)}\right]}{B(A-2B)(1+|b_1|r)^2} \le |f|\\
&\le -|b_1|\frac{\left(\frac{|b_1|}{B}\right)^{1-\frac{A}{B}}{}_2F_1\left[2-\frac{A}{B}, 2-\frac{A}{B}, 3-\frac{A}{B}, 1-\frac{|b_1|}{B}\right]}{A-2B}\\
&+|b_1|\frac{1}{2}(1+A)r^2F_1\left[2, 2-\frac{A}{B}, 1, 3, -Br, -|b_1|r\right]\\
&+|b_1|\frac{1}{3}Ar^3F_1\left[3, 2-\frac{A}{B}, 1, 4, -Br, -|b_1|r\right]\\
&+|b_1|\frac{|b_1|(1-Br)^{\frac{A}{B}}\left(\frac{|b_1|(1+Br)}{B(1+|b_1|r)}\right)^{-\frac{A}{B}}{}_2F_1\left[2-\frac{A}{B}, 2-\frac{A}{B}, 3-\frac{A}{B}, \frac{B-|b_1|}{B(1+|b_1|r)}\right]}{B(A-2B)(1+|b_1|r)^2},
\end{aligned} \tag{22}$$

for all $|z| = r < 1$*, where* ${}_2F_1$ *and* F_1 *are denote the Gauss and Appel hypergeometric functions, respectively [1].*

Proof. Using Lemma 2.2 and the inequalities in (18), after the simple calculations, we get

$$(|h'(z)| - |g'(z)|)|dz| \le |df| \le (|h'(z)| + |g'(z)|)|dz| \Rightarrow$$

$$|h'(z)|(1 - w(z))|dz| \le |df| \le |h'(z)|(1 + w(z))|dz|$$

$$\begin{aligned}
|b_1|(1-Ar)(1-Br)^{\frac{A-2B}{B}}\frac{(1-|b_1|)(1-r)}{1+|b_1|r}dr \le |df|\\
\le |b_1|(1+Ar)(1+Br)^{\frac{A-2B}{B}}\frac{(1+|b_1|)(1+r)}{1+|b_1|r}dr
\end{aligned} \tag{23}$$

for all $|z| = r < 1$. Integrating the inequality (23), we obtain (22).

Theorem 2.6 *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}^*_{\mathcal{HPST}}(A,B)$, then*

$$|b_{n+1}| \leq \frac{|b_1|}{n+1} \sum_{k=1}^{n+1} k(A-B) \prod_{m=0}^{k-2} \frac{(A-B)+mB}{m+1}. \tag{24}$$

Proof. Since $f = h(z) + \overline{g(z)} \in \mathcal{S}^*_{\mathcal{HPST}}(A,B)$, then we have

$$p(z) = \frac{g'(z)}{b_1 h'(z)},\ p(z) \in \mathcal{P}(A,B).$$

Therefore, we can write

$$1 + p_1 z + p_2 z^2 + \cdots + p_n z^n + \cdots = \frac{b_1 + 2b_2 z + \cdots}{b_1(1 + 2a_2 z + \cdots)} \Rightarrow$$

$$(1 + p_1 z + p_2 z^2 + \cdots + p_n z^n + \cdots)(b_1(1 + 2a_2 z + \cdots)) = b_1 + 2b_2 z + \cdots \Rightarrow$$

$$b_{n+1} = \frac{b_1}{n+1} \sum_{k=1}^{n+1} k a_k p_{n-k+1} \tag{25}$$

where $a_1 \equiv 1$, $p_0 \equiv 1$. Using Theorem 1.2 in (25) we obtain (24).

References

[1] Abramowitz, M., Stegun, I., Handbook of Mathematical Functions, Dover Pub., New York, 1970.

[2] Duren, P., Harmonic mappings in the plane. Cambridge Tracts in Mathematics, 156. Cambridge University Press, Cambridge, 2004.

[3] Jack, I.S., Functions starlike and convex of order α, J. London Math. Soc. (2) 3 (1971), 469-474.

[4] Janowski, W., Some extremal problems for certain families of analytic functions, I. Ann. Polon. Math. 28 (1973), 297-326.

[5] Libera, R.J., Some classes of regular univalent functions, Proc. Amer. Math. Soc. 16 (1965) 755-758.

[6] Polatoğlu, Y., Bolcal, M., A coefficient inequality for the class of analytic functions in the unit disc, Int. J. Math. Math. Sci. 59 (2003), 3753-3759.

Received: September, 2013