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Some cardinal properties of complete linked systems with compact elements and absolute regular spaces

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Abstract

In the paper, we study cardinal properties of the space of complete linked systems containing compact elements and absolute regular space.

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1 Introduction

In the paper, we study the density, net weight, π-weight, the Souslin number, weakly density of the space of complete linked systems with compact elements. We also consider absolute regular spaces. If two absolute spaces X and Y are co-absolute, they have common properties, as, for example, $c(X) = c(Y)$, $d(X) = d(Y)$, $\pi w(X) = \pi w(Y)$, and if one from these spaces is compact, finally compact, paracompact, locally compact, complete in the Čech sense, respectively, then the other space is the same [1].

2 Preliminary Notes

A system $\xi = \{F_\alpha : \alpha \in A\}$ of closed subsets of a space X is called *linked* if any two elements from ξ intersect. Any linked system can be complemented to a maximal linked system (MLS), but this complement is, as a rule, not unique [2].

Proposition 2.1 . *[2]. A linked system ξ of a space X is a MLS iff it possesses the following completeness property:*
if a closed set $A \subset X$ intersects with any element form ξ, then $A \in \xi$.

Denote by λX the set of all MLS of the space X. For a closed set $A \subset X$, put

$$A^{+} = \{\xi \in \lambda X : A \in \xi\}.$$

For an open set $U \subset X$, set

$$O(U) = \{\xi \in \lambda X : \text{ there is an } F \in \xi \text{ such that } F \subset U\}.$$

The family of subsets in the form of $O(U)$ covers the set λX $(O(X) = \lambda X)$, that's why it forms an open subbase of the topology on λX. The set λX equipped with this topology is called *the superextension* of X.

A.V. Ivanov [3] defined the space NX of complete linked systems (CLS) of a space X in a following way:

Definition 2.2 . *[3]. A linked system M of closed subsets of a compact X is called a complete linked system (CLS) if for any closed set of X, the condition*
"Any neighborhood OF of the set F consists of a set $\Phi \in M$"
implies $F \in M$.

A set NX of all complete linked systems of a compact X is called *the space NX of CLS of X*. This space is equipped with the topology, the open basis of which is formed by sets in the form of
$E = O(U_1, U_2, \ldots, U_n)\langle V_1, V_2, \ldots, V_s\rangle = \{M \in NX :$ for any $i = 1, 2, \ldots, n$ there exists $F_i \in M$ such that $F_i \subset U_i$, and for any $j = 1, 2, \ldots, s$, $F \cap V_j \neq \emptyset$ for any $F \in M\}$, where $U_1, U_2, \ldots, U_n$, $V_1, V_2, \ldots, V_s$ are nonempty open in X sets [3].

Definition 2.3 . *[4]. Let M be a complete linked system of a compact X. The CLS M will be said a thin complete linked system if M contains at least one finite element.*

We denote a thin complete linked system M by a TCLS.

Definition 2.4 . *[4]. We call an N-thin kernel of a topological space X the space*

$$N^*X = \{M \in NX : \ M \ \text{ is a TCLS}\}.$$

Definition 2.5 . *[1]. A continuous mapping $f : X \to Y$ is called perfect if X is a Hausdorff space, f is a closed mapping and all preimages $f^{-1}(y)$ are compact spaces for each $y \in Y$.*

Proposition 2.6 . *[4]. Let $\mu = \{\Phi_1, \Phi_2, \dots, \Phi_n\}$ be a finite linked system of closed subsets of a space X. Then the system*

$$M = \{F \in \exp X : \exists \Phi_i \in \mu, \ \Phi_i \subset F\}$$

is a complete linked system of X.

Definition 2.7 . *[1]. A continuous mapping $f : X \to Y$ of a space X onto a space Y is called irreducible if $f(A) \neq Y$ for any proper closed subset A of the space X.*

Definition 2.8 . *[1]. A continuous mapping $f : X \to Y$ is called separated if for any $x_1 \neq x_2 \in X$, satisfying the condition $f(x_1) = f(x_2)$, there exist disjunct in X neighborhoods.*

Definition 2.9 . *[1]. Let X be a topological space. An extremally disconnected space qX is called an absolute of X if there exists a perfect irreducible continuous mapping $\pi_X : qX \xrightarrow{onto} X$.*

Definition 2.10 . *[5]. We say that the weakly density of a topological space X is equal to $\tau \geq \aleph_0$ if τ is the least cardinal number such that there is in X a π-base decomposing on τ centered systems of open sets, i.e. $B = \cup\{B_\alpha : \alpha \in A\}$ is a π-base where B_α is a centered system of open sets for any $\alpha \in A$, $|A| = \tau$.*

The weakly density of a topological space X is denoted by $wd(X)$. If $wd(X) = \aleph_0$, then the topological space X is called weakly separable [6].

Theorem 2.11 . *[6]. Two regular spaces X and Y are co-absolute iff there exists a regular space Z and perfect irreducible mappings $p_X : Z \to X$ and $p_Y : Z \to Y$ such that $p_X(Z) = X$, $p_Y(Z) = Y$.*

Proposition 2.12 . *[5]. If Y is everywhere dense in X, then $wd(X) = wd(Y)$.*

3 Main Results

Obviously, any MLS ξ is a CLS, hence, $\lambda X \subset NX$.

Let's cite an example of a linked system which is a CLS and is not a MLS.

Example 3.1 . *Let $X = [-10, 10]$. For a finite linked system $\xi = \{[0,1], [1,3]\}$, put $M = \{F \in \exp X : [0,2] \subset F \text{ or } [1,3] \subset F\}$. We show that M is a CLS, but it is not a MLS.*

Suppose that the system M is not a CLS. Then there is a closed in X subset F_0, any neighborhood OF_0 of which contains any element $F \in M$ and $F_0 \notin M$. Since $F_0 \notin M$, we have, by constructing the system M, $[0;2] \not\subset F_0$ and $[1;3] \notin F_0$. Now let's show that $[0;2] \cap F_0 \neq \emptyset$ and $[1;3] \cap F_0 \neq \emptyset$. Suppose that there exists a set $[0;2] \in \mu$ such that $[0;2] \cap F_0 = \emptyset$. By virtue of normality of the space X, there is a neighborhood OF_0 of the set F_0 such that $[0;2] \cap OF_0 = \emptyset$. By supposition, the neighborhood OF_0 contains an element $F \in M$, hence, there will be the set $[1;3] \in \mu$ such that $[1;3] \subset F \subset OF_0$, hence, $[0;2] \cap [1;3] = \emptyset$. The obtained contradiction proves that $[0;2] \cap F_0 \neq \emptyset$ and $[1;3] \cap F_0 \neq \emptyset$. Thus, we have $[0;2] \setminus F_0 \neq \emptyset$ and $[1;3] \setminus F_0 \neq \emptyset$. Take a point x_1 and x_2 from each set $[0;2] \setminus F_0 \neq \emptyset$ and $[1;3] \setminus F_0 \neq \emptyset$, respectively, then we get the set $\mu = \{x_1, x_2\}$. Obviously, the set $O'F_0 = X \setminus \mu$ is a neighborhood of F_0, and it is clear, $O'F_0$ does not contain any element F from M, what contradicts to the fact that any neighborhood of the set F_0 contains an element from the system M. The obtained contradiction proves that the system M is a CLS of the topological space X, but it is not a MLS since $[0,2] \cap [1,3] = [1,2] \notin M$.

Definition 3.2 . *Let M be a complete linked system of a topological space X. A CLS M is called a compact complete linked system if M contains at least one compact element.*

We denote a compact complete linked system M by a CCLS.

Definition 3.3 . *Let X be a topological space. The space*

$$N_c X = \{M \in NX : \ M \text{ is a } CCLS\}$$

will be said the compact kernel of X.

Definition 3.4 . *Let M be a complete linked system of a topological space X. A CLS M will be said a metrizable compact complete linked system if M contains at least one metrizable compact element.*

We denote a metrizable compact complete linked system M by a MCCLS.

Definition 3.5 . *Let X be a topological space. A space*

$$N_{cm}X = \{M \in NX : \ M \text{ is a MCCLS}\}$$

will be said the metrizable compact kernel of X.

It is clear, $N^*X \subseteq N_{cm}X \subseteq N_cX \subseteq NX$ for any topological space X.
Obviously, if X is a discrete space, then $N^*X = N_cX$, and $N_cX = NX$ for a compact space X.

Theorem 3.6 . *Let X be a topological T_1-space. Then:*
1) $\pi w(N^*X) = \pi w(N_{cm}X) = \pi w(N_cX) = \pi w(X)$;
2) $d(X) = d(N^*X) = d(N_{cm}X)$;
3) $n\pi w(N^*X) = n\pi w(N_{cm}X) = n\pi w(X)$;
4) If X is an infinite Tychonoff space, then

$$c(N^*X) = c(N_cX) = c(N_{cm}X) = c(NX) = Sup\left\{c\left(X^n\right) : n \in N\right\}.$$

5) If X is an infinite Tychonoff space, then

$$wd(N^*X) = wd(N_cX) = wd(N_{cm}X) = wd(NX) \le wd(X).$$

Proof. 1) In [4], it was proved that $[N^*X]_{NX} = NX$, what implies immediately $[N_{cm}X]_{NX} = [N_cX]_{NX} = NX$. It is known, π-weights of everywhere dense subsets coincide. Hence, we have $\pi w(N^*X) = \pi w(N_{cm}X) = \pi w(N_cX) = \pi w(X)$.

2) a) We show that $d\left(N_{cm}X\right) \le d(X)$. Let X_0 be an everywhere dense in X subset such that $|X_0| = d(X) = \tau$. Set:

$$\exp_\omega(X_0, X) = \{\Phi \in \exp X : \ \Phi \subset X_0 \text{ and } |\Phi| \le \omega\},$$

and

$$\Sigma(X_0, X) = \{\mu \subset \exp(X_0, X) : \ \mu \text{ is a MCCLS}\}.$$

Now let $\mu \in \Sigma(X_0, X)$. Put $M_\mu = \{F \in \exp X : \exists \Phi \in \mu : \Phi \subset F\}$. Obviously, the set $N_{cm}(X_0, X) = \{M_\mu : \mu \in \Sigma(X_0, X)\}$ has the power equal to $\tau = d(X)$, and by Proposition 2.6 [4], we have $N_{cm}(X_0, X) \subset N_{cm}X$. Let's show that a set in $N_{cm}(X_0, X)$ is an everywhere dense set in $N_{cm}X$. Let

$E = O(U_1, U_2, \ldots, U_n)\langle V_1, V_2, \ldots, V_s\rangle$ be arbitrary nonempty open base element in $N_{cm}X$. Recall that the system

$$S(E) = \{U_i \cap V_j : i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, s\} \cup S(O)$$

of open in X subsets is called *the pairwise trace of the base element E in X*. Here $S(O)$ is the pairwise trace of the element $O(U_1, U_2, \ldots, U_n)$ in X. Let $S(E) = \{W_1, W_2, \ldots, W_n\}$ be the pairwise trace of the element E in X. Since X_0 is an everywhere dense set in X, we have $X_0 \cap W_i \neq \emptyset$ where $i = 1, 2, \ldots, m$. Now take a point x_i from each intersection $X_0 \cap W_i \neq \emptyset$, then we obtain the set $\sigma = \{x_1, x_2, x_m\}$. Set $\Phi_j = \{x_j \in \sigma : x_j \in U_i\}$, $i = 1, 2, \ldots, n$. Obviously, the system $\mu = \{\Phi_1, \Phi_2, \ldots, \Phi_n\}$ is a linked system of sets closed in X, and it is clear that $\mu \in \Sigma(X_0, X)$. Hence, $M_\mu \in N(X_0, X)$. By constructing the system M_μ, $\Phi \cap V_j \neq \emptyset$ for any $F \in M_\mu$ and for each $j = 1, 2, \ldots, s$. Moreover, it is clear, $\Phi_i \subset U_i$ where $i = 1, 2, \ldots, n$. Hence, we get $M_\mu \in E$. Thus, $N_{cm}(X_0, X) \cap E \neq \emptyset$. By virtue of arbitrariness of a base element E in $N_{cm}X$, the set $N_{cm}(X_0, X)$ intersects with any set open in $N_{cm}X$. Hence, $N_{cm}(X_0, X)$ is a set everywhere dense in $N_{cm}X$. Since $|N_{cm}(X_0, X)| = |X_0| = d(X)$, we have $d(N_{cm}X) \leq d(X)$.

b) Now we show that $d(X) \leq d(N_{cm}X)$. Let $B = \{M_i : i \in \theta\}$ be a set everywhere dense in $N_{cm}X$ such that $|B| = d(N_{cm}X) = \tau$. Fix a countable set F_i from any MCCLS $M_i \in B$. Then we obtain the system $H = \{F_i : i \in \theta\}$. Put $X_0 = \cup H$. It is clear, $|X_0| = \tau$. Let's show that X_0 is everywhere dense in X. Let U be arbitrary nonempty set open in X. Then the set $E = O(U)(X)$ is a set open in NX, hence, the set $E_{cm} = O(U)(X) \cap N_{cm}X$ is a set open in $N_{cm}X$. Then there will be at least an element $M_i \in B$ such that $M_i \in E_{cm}$, hence $F_i \cap U \neq \emptyset$, moreover $X_0 \cap U \neq \emptyset$. Thus, by virtue of arbitrariness of the set U in X, the set X_0 intersects with any set open in X, i.e. $[X_0] = X$. Therefore $d(X) \leq |X_0| = \tau = d(N_{cm}X)$. We obtain from a) and b) the equality $d(N_{cm}X) = d(X)$.

The equality $d(X) = d(N^*X) = d(N_{cm}X)$ follows from a) and b).

3) The proof of the equality 3) follows immediately from the equality 2).

4) It is known, the Souslin numbers of everywhere dense subsets are equal. It is clear, the sets $N_{cm}(X)$ and $N_c(X)$ are everywhere dense in $N(X)$. Then $c(N^*X) = c(N_cX) = c(N_{cm}X) = c(NX) = Sup\{c(X^n) : n \in N\}$.

5) The equality $wd(N^*X) = wd(N_cX) = wd(N_{cm}X) = wd(NX)$ follows from Proposition 2.12 [5]. Lets' show correctness of the inequality

$$wd(NX) \leq wd(X).$$

Let $wd(X) = \tau \geq \aleph_0$. Then $d(bX) = wd(X) = \tau$ for any compactification bX of the space X. We obtain from [4] that $d(N(bX)) \leq d(bX) = \tau$. Then $wd(NX) \leq wd(X) = \tau$.

Proposition 3.7 . *Let Y is everywhere dense in X. Then N^*Y is everywhere dense in NX.*

Proof. Let $E = O(U_1, U_2, \dots, U_n)\langle V_1, V_2, \dots, V_s\rangle$ be arbitrary nonempty open base element in NX. Consider the pairwise trace of E in X, i.e. consider the following system of subsets open in X :

$$S(E) = \{U_i \cap V_j : i = 1, 2, \dots, n, j = 1, 2, \dots, s\} \cup S(O)$$

where $S(O)$ is the pairwise trace of the element $O(U_1, U_2, \dots, U_n)$ in X. Let $S(E) = \{W_1, \dots, W_p\}$ be the pairwise trace of the element E in X. Since Y is a set everywhere dense in X, we have $Y \cap W_i \neq \emptyset$ where $i = 1, 2, \dots, p$. Fix a point x_i in each intersection in the form of $Y \cap W_i$. Then we obtain the set $\sigma = \{x_1, x_2, \dots, x_p\}$. Set $\Phi_i = \{x_j \in \sigma : x_j \in W_i\}$ where $i = 1, 2, \dots, n$. It is clear, the system $\mu = \{\Phi_1, \Phi_2, \dots, \Phi_n\}$ is a linked system of the space X. Put $M_\mu = \{F \in \exp X : \exists \Phi_i \in \mu : \Phi_i \subset F\}$. By constructing the system M_μ, we have $F \cap V_j \neq \emptyset$ for any $F \in M_\mu$ and for each $j = 1, 2, \dots, s$. Moreover, it is clear, $\Phi_i \subset U_i$ where $i = 1, 2, \dots, n$. Then by Proposition 2.6 [4] we have that M_μ is a CLS and, it is obvious, M_μ is a TCLS, and $M_\mu \in E$. By virtue of arbitrariness of an open base element E in NX, the set N^*Y intersects with any set open in NX. Therefore N^*Y is everywhere dense in NX.

Corollary 3.8 . *Let Y be everywhere dense in X. Then NY is everywhere dense in NX.*

Theorem 3.9 . *Let X, Y be arbitrary topological T_1-spaces and there be a continuous, closed and irreducible mapping "onto" $f : X \to Y$. Then $wd(X) = wd(Y)$.*

Proof. At first, we show correctness of the inequality $wd(Y) \leq wd(X)$. Let $wd(X) = \tau \geq \aleph_0$, i.e. there be a π-base in X $B = \cup\{B_\alpha : \alpha \in A, |A| = \tau\}$ where $B_\alpha = \{U_s^\alpha : s \in A_\alpha\}$ is a centered system of open sets for each $\alpha \in A$. Let's show that $wd(Y) \leq \tau$. Put $B'_\alpha = \{f(U_s^\alpha) : s \in A_\alpha\}$ and

$B' = \cup\{B'_\alpha : \alpha \in A\}$. Let's show that each system B'_α is centered for any $\alpha \in A$. Choose arbitrary finite sets $f\left(U_{s_1}^\alpha\right) \in B'_\alpha, \dots, f\left(U_{s_k}^\alpha\right) \in B'_\alpha$. Consider preimages of these sets $f^{-1}\left(f\left(U_{s_i}^\alpha\right)\right)$, $i = 1, \dots, k$. It is clear, $U_{s_i}^\alpha \subset f^{-1}\left(f\left(U_{s_1}^\alpha\right)\right)$, $i = 1, \dots, k$. By virtue of centrality of the system B_α, we have $\bigcap\limits_{i=1}^{k} U_{s_i}^\alpha \neq \emptyset$, hence $\emptyset \neq f\left(\bigcap\limits_{i=1}^{k} U_{s_i}^\alpha\right) \subset \bigcap\limits_{i=1}^{k} f\left(U_{s_i}^\alpha\right)$. It means that the system B'_α is centered for each $\alpha \in A$.

Let's show that the system B'_α is a π-net in Y. Let G be arbitrary nonempty open set in X. The system B is a π-base in X, that's why there exists $U_s^\alpha \in$

$B_\alpha \subset B$ such that $U_s^\alpha \subset f^{-1}(G)$. Hence, $f(U_s^\alpha) \subset G$. Therefore $f(U_s^\alpha) \in B'_\alpha$. It means that $wd(Y) \leq \tau$.

Let's show the inverse, i.e. $wd(X) \leq wd(Y)$. Let $wd(Y) = \tau \geq \aleph_0$, i.e. there exists a π-base $B = \cup\{B_\alpha : \alpha \in A, |A| = \tau\}$ in Y where $B_\alpha = \{U_s^\alpha : s \in A_\alpha\}$ is a centered system of open sets for each $\alpha \in A$. It is clear, $f^{-1}(B_\alpha) = \{f^{-1}(U_s^\alpha) : s \in A_\alpha\}$ is centered for each $\alpha \in A$. Really, for any finite set $U_{s_1}^\alpha \in B_\alpha, \ldots, U_{s_k}^\alpha \in B_\alpha$ in Y, we have

$$\cap\left\{f^{-1}\left(U_{s_i}^\alpha\right) : i = 1, \ldots, k\right\} = f^{-1}\left(\cap\left\{U_{s_i}^\alpha : i = 1, \ldots, k\right\}\right).$$

Since B_α is centered, we have that $\cap\left\{U_{s_i}^\alpha : i = 1, \ldots, k\right\} \neq \emptyset$.

Therefore $\cap\left\{f^{-1}\left(U_{s_i}^\alpha\right) : i = 1, \ldots, k\right\} \neq \emptyset$.

Set $f^{-1}(B) = \cup\{f^{-1}(B_\alpha) : \alpha \in A\}$ and show that the system $f^{-1}(B)$ is a π-base in X. Let $G \subset X$ be arbitrary nonempty open subset in X. If $G = X$, then for arbitrary $V \in B_\alpha$, $\alpha \in A$, we have $f^{-1}(V) \subset G$.

Let now $G \neq X$. Then $X \setminus G$ is a nonempty closed subset in X. By virtue of closeness and irreducibility of the mapping f, we obtain that $f(X \setminus G)$ is a closed subset in Y, and $f(X \setminus G) \neq Y$. It follows from here that $Y \setminus f(X \setminus G)$ is a nonempty open subset in Y. Therefore there is α and $U_\alpha \in B_\alpha \subset B$ such that $U_\alpha \subset Y \setminus f(X \setminus G)$. Hence, $f^{-1}(U_\alpha) \subset f^{-1}(Y \setminus f(X \setminus G)) = X \setminus f^{-1}(f(X \setminus G)) \subset X \setminus (X \setminus G) = G$, i.e. $f^{-1}(U_\alpha) \subset G$. It means that the system $f^{-1}(B)$ is a π-base in X. Thus, $wd(X) \leq wd(Y)$.

Theorem 3.10 . *Let regular spaces X and Y be co-absolute. Then*

$$wd(X) = wd(Y).$$

Proof. If X and Y are co-absolute, then we consider their common absolute Z. By definition of the absolute, there are perfect irreducible mappings $p_X : Z \to X$ and $p_Y : Z \to Y$. If $wd(X) = \tau \geq \aleph_0$, then $wd(Z) \leq \tau$. By virtue of Theorem 3.9, $wd(Y) \leq \tau$. Conversely, let $wd(Y) = \tau \geq \aleph_0$. Then $wd(Z) \leq \tau$. By Theorem 3.9, $wd(X) \leq \tau$.

Corollary 3.11 . *Let regular spaces X and Y be co-absolute. The space X is weakly separable iff Y is weakly separable.*

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