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Pairs of implications induced by pseudo t-conorms

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Abstract

In this paper, we construct pseudo t-norms and pairs of implications induced by pseudo t-norms and pairs of negations. Moreover, we investigate their properties and give examples.

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pseudo t-norms, pseudo t-conorms, pairs of negations, pairs of implications

1 Introduction

Georgescu and Popescue [1-4] introduced pseudo t-norms and generalized residuated lattices in a sense as non-commutative property. This concept provides tools for pseudo BL-algebras and pseudo MV-algebras. Kim [7] introduced pairs of (interval) negations and (interval) implications. which are induced by non-commutative property. Let $(L, \wedge, \vee, \odot, \rightarrow, \Rightarrow, \top, \perp)$ be a complete generalized residuated lattice with the law of double negation defined as $a = n_1(n_2(a)) = n_2(n_1(a))$ where $n_1(a) = a \Rightarrow \perp$ and $n_2(a) = a \rightarrow \perp$ (ref. [1-4,8]). We consider a pair of two implications defined by $a \Rightarrow b = \bigvee\{c \mid a \odot c \leq b\}$ and $a \rightarrow b = \bigvee\{c \mid c \odot a \leq b\}$. Moreover, we consider a pair of two negations defined by $a \Rightarrow \perp$ and $a \rightarrow \perp$.

In this paper, we construct pseudo t-norms and pairs of implications induced by pseudo t-norms and pairs of negations. Moreover, we investigate their properties and give examples.

2 Preliminaries

In this paper, we assume that $(L, \vee, \wedge, \perp, \top)$ is a bounded lattice with a bottom element $\perp$ and a top element $\top$. Moreover, we define the following definitions in a sense as non-commutative [1-4, 7].

Definition 2.1 [1,2] A map $T : L \times L \to L$ is called a *pseudo t-norm* if it satisfies the following conditions:

(T1) $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in L$,

(T2) If $y \le z$, $T(x, y) \le T(x, z)$ and $T(y, x) \le T(z, x)$,

(T3) $T(x, \top) = T(\top, x) = x$.

A pseudo t-norm is called a *t-norm* if $T(x, y) = T(y, x)$ for $x, y \in L$

A map $S : L \times L \to L$ is called a *pseudo t-conorm* if it satisfies the following conditions:

(S1) $S(x, S(y, z)) = S(S(x, y), z)$ for all $x, y, z \in L$,

(S2) If $y \le z$, $S(x, y) \le S(x, z)$ and $S(y, x) \le S(z, x)$,

(S3) $S(x, \bot) = S(\bot, x) = x$.

A pseudo t-conorm is called a *t-conorm* if $S(x, y) = S(y, x)$ for $x, y \in L$.

Definition 2.2 [7] A pair (n_1, n_2) with maps $n_i : L \to L$ is called a *pair of negations* if it satisfies the following conditions:

(N1) $n_i(\top) = \bot, n_i(\bot) = \top$ for all $i \in \{1, 2\}$.

(N2) $n_i(x) \ge n_i(y)$ for $x \le y$ and $i \in \{1, 2\}$.

(N3) $n_1(n_2(x)) = n_2(n_1(x)) = x$ for all $x \in L$.

Definition 2.3 [7] A pair (I_1, I_2) with maps $I_1, I_2 : L \times L \to L$ is called a *pair of implications* if it satisfies the following conditions:

(I1) $I_i(\top, \top) = I_i(\bot, \top) = I_i(\bot, \bot) = \top, I_i(\top, \bot) = \bot$ for all $i \in \{1, 2\}$.

(I2) If $x \le y$, then $I_i(x, z) \ge I_i(y, z)$ for all $i \in \{1, 2\}$.

(I3) $I_i(\top, x) = x$ for all $x \in L$ and $i \in \{1, 2\}$.

A pair (I_1, I_2) of implications is called a *pair of E-implications* if it satisfies the following exchange properties:

(E) $I_1(x, I_2(y, z)) = I_2(y, I_1(x, z))$ for all $x, y, z \in L$.

A pair (I_1, I_2) of implications is called a *pair of S-implications* if it satisfies the following strong properties:

(S) $I_1(I_2(x, \bot), \bot) = I_2(I_1(x, \bot), \bot) = x$.

A pair (I_1, I_2) of implications is called a *pair of SE-implications* if it satisfies conditions (E) and (S).

3 Pairs of implications induced by pseudo t-conorms

Theorem 3.1 *Let $(L, \vee, \wedge, \top, \bot)$ be a bounded lattice, $S : L \times L \to L$ be a pseudo t-conorm and (n_1, n_2) a pair of negations. We define $S^t, T_{12}, T_{21}, T^t_{12}, T^t_{21} : L \times L \to L$*

$$S^t(x, y) = S(y, x),$$

$$T_{12}(x,y) = n_1(S(n_2(x), n_2(y)))$$
$$T_{21}(x,y) = n_2(S(n_1(x), n_1(y)))$$
$$T_k^t(x,y) = T_k(y,x),\ k \in \{12, 21\}$$

The the following properties hold.
(1) S^t *is a pseudo t-conorms.*
(2) $T_{12}, T_{21}, T^t_{12}, T^t_{21}$ *are pseudo t-norms.*
(3) $T_{12} = T_{21}$ *iff* $T^t_{12} = T^t_{21}$ *iff*

$$S(x,y) = n_2n_2(S(n_1(n_1(x)), n_1(n_1(x))))$$

(4) *If* $n_1 = n_2$*, then* $T_{12} = T_{21}$ *and* $T^t_{12} = T^t_{21}$.

Proof (1) (S1) $S^t(S^t(x,y),z) = S^t(x, S^t(y,z))$ from

$$\begin{aligned} &S^t(S^t(x,y),z) = S^t(S(y,x),z) = S(z, S(y,x)),\\ &= S^t(x, S^t(y,z)) = S^t(x, S(z,y)) = S(S(z,y),x). \end{aligned}$$

(S2) $S^t(x, \perp) = S(\perp, x) = x$. Similarly, $S^t(\perp, x) = S(x, \perp) = x$.
(S3) If $x \le z$ and $y \le w$, then

$$S^t(x,y) = S(y,x) \le S(w,z) = S^t(z,w).$$

Hence S^t is a pseudo t-conorms.
(2) (T1) $T_{12}(T_{12}(x,y),z) = T_{12}(x, T_{12}(y,z))$ from

$$\begin{aligned} &T_{12}(T_{12}(x,y),z)\\ &= n_1(S(n_2T_{12}(x,y)), n_2(z)))\\ &= n_1(S(n_2(n_1(S(n_2(x), n_2(y))))), n_2(z)))\\ &= n_1(S(S(n_2(x), n_2(y)), n_2(z)))\\ &= n_1(S(n_2(x), S(n_2(y), n_2(z)))),\\ &T_{12}(x, T_{12}(y,z))\\ &= n_1(S(n_2(x), n_2(T_{12}(y,z))))\\ &= n_1(S(n_2(x), n_2(n_1(S(n_2(y), n_2(z))))))\\ &= n_1(S(n_2(x), S(n_2(y), n_2(z)))). \end{aligned}$$

(T2) $T_{12}(x, \top) = n_1(S(n_2(x), n_2(\top)) = n_1(n_2(x)) = x$ and $T_{12}(\top, x) = x$.
(T3) If $x \le z$ and $y \le w$, then $T(x,y) \le T(z,w)$.
Hence T_{12} is a pseudo t-conorm. Similarly, $T_{21}, T^t_{12}, T^t_{21}$ are pseudo t-norms.
(3)

$$\begin{aligned} &T_{12}(x,y) = \mathcal{T}_{21}(x,y)\\ &\text{iff } n_1(S(n_2(x), n_2(y))) = n_2(S(n_1(x), n_1(y)))\\ &\text{iff } S(n_2(x), n_2(y)) = n_2(n_2(S(n_1(x), n_1(y))))\\ &\text{iff } S(x,y) = n_2(n_2(S(n_1(n_1(x)), n_1(n_1(y)))))\\ &\text{iff } T^t_{12}(x,y) = \mathcal{T}^t_{21}(x,y). \end{aligned}$$

(4) By (3), since $n_2 \circ n_2 = n_1 \circ n_1 = id_L$, it is trivial.

Example 3.2 Put $L = \{(x,y) \in R^2 \mid (0,1) \le (x,y) \le (2,3)\}$ where $(0,1)$ is the bottom element and $(2,3)$ is the top element where

$$(x_1,y_1) \le (x_2,y_2) \Leftrightarrow y_1 < y_2 \text{ or } y_1 = y_2, x_1 \le x_2.$$

(1) A map $S : L \times L \to L$ is defined as

$$S((x_1,y_1),(x_2,y_2)) = (x_2 + x_1y_2, y_1y_2) \wedge (2,3).$$

(S1) $S(S((x_1,y_1),(x_2,y_2)),(x_3,y_3)) = S((x_1,y_1),S((x_2,y_2),(x_3,y_3)))$ from:

$$\begin{aligned}
&S(S((x_1,y_1),(x_2,y_2)),(x_3,y_3))\\
&= S((x_2 + x_1y_2, y_1y_2) \wedge (2,3),(x_3,y_3))\\
&= (x_3 + x_2y_3 + x_1y_2y_3, y_1y_2y_3) \wedge (2,3).\\
&S((x_1,y_1),S((x_2,y_2),(x_3,y_3)))\\
&= S((x_1,y_1),(x_3 + x_2y_3, y_2y_3) \wedge (2,3))\\
&= (x_3 + x_2y_3 + x_1y_2y_3, y_1y_2y_3) \wedge (2,3).
\end{aligned}$$

(S2) If $(x_1,y_1) \le (x_2,y_2)$, then $y_1 < y_2$ or $y_1 = y_2, x_1 \le x_2$. Thus

$$\begin{aligned}
&S((x_1,y_1),(x_3,y_3)) = (x_3 + x_1y_3, y_1y_3) \wedge (2,3)\\
&\le (x_3 + x_2y_3, y_2y_3) \wedge (2,3) = S((x_2,y_2),(x_3,y_3)).
\end{aligned}$$

(S3)

$$S((x_1,y_1),(0,1)) = (x_1,y_1) = S((0,1),(x_1,y_1)).$$

Then S is a pseudo t-conorm but not t-conorm because

$$(2,2) = S((-1,2),(3,1)) \ne S((3,1),(-1,2)) = (5,2).$$

(2) We define a pair (n_1,n_2) as follows

$$n_1(x,y) = (2 - \frac{3x}{y}, \frac{3}{y}), \ \ n_2(x,y) = (\frac{2-x}{y}, \frac{3}{y}).$$

Then (n_1,n_2) is a pair of negations from:

$$n_1(n_2(x,y)) = (x,y), \ \ n_2(n_1(x,y)) = (x,y).$$

(3) By Theorem 3.1(2), we obtain

$$\begin{aligned}
&T_{12}((x_1,y_1),(x_2,y_2)) = n_1S(n_2(x_1,y_1),n_2(x_2,y_2)))\\
&= n_1S((\tfrac{2-x_1}{y_1}, \tfrac{3}{y_1}),(\tfrac{2-x_2}{y_2}, \tfrac{3}{y_2})) = n_1(\tfrac{2-x_2}{y_2} + \tfrac{3(2-x_1)}{y_1y_2}, \tfrac{9}{y_1y_2})) \vee (0,1)\\
&= (x_1 - \tfrac{2}{3}y_1 + \tfrac{1}{3}x_2y_1, \tfrac{1}{3}y_1y_2) \vee (0,1),
\end{aligned}$$

$$\begin{aligned}
&T_{21}((x_1,y_1),(x_2,y_2)) = n_2S(n_1(x_1,y_1),n_1(x_2,y_2)))\\
&= n_2S((2 - \tfrac{3x_1}{y_1}, \tfrac{3}{y_1}),(2 - \tfrac{3x_2}{y_2}, \tfrac{3}{y_2})) = n_2(2 - \tfrac{3x_2}{y_2} - (2 - \tfrac{3x_1}{y_1})\tfrac{3}{y_2}, \tfrac{3}{y_1y_2})) \vee (0,1)\\
&= (x_1 - \tfrac{2}{3}y_1 + \tfrac{1}{3}x_2y_1, \tfrac{1}{3}y_1y_2) \vee (0,1).
\end{aligned}$$

(4) Since

$$n_1(n_1(x,y)) = (3x - 2y + 2, y), \quad n_2(n_2(x,y)) = (\frac{x+2y-2}{3}, y)$$

$$\begin{aligned}
&n_2(n_2(T(n_1(n_1(x_1,y_1)), n_1(n_1(x_2,y_2)))))\\
&= n_2(n_2(T((3x_1 - 2y_1 + 2, y_1), (3x_2 - 2y_2 + 2, y_2))),\\
&= n_2(n_2((\tfrac{1}{3}(3x_1 - 2y_1 + x_2y_1), \tfrac{1}{3}y_1y_2))) \vee (0,1),\\
&= (x_1 - \tfrac{2}{3}y_1 + \tfrac{1}{3}x_2y_1, \tfrac{1}{3}y_1y_2) \vee (0,1)\\
&= T((x_1,y_1),(x_2,y_2)).
\end{aligned}$$

By Theorem 3.1 (3), $T_{12} = T_{21}$ and $T_{12}^t = T_{21}^t$.

Theorem 3.3 *Let $(L, \vee, \wedge, \top, \bot)$ be a bounded lattice, $S : L \times L \to L$ be a pseudo t-conorm and (n_1, n_2) a pair of negations. For $i = \{1, ..., 4\}$, we define $I_i : L \times L \to L$ as follows;*

$$I_1(x,y) = S(n_1(x), y), \quad I_2(x,y) = S(y, n_2(x)),$$

$$I_3(x,y) = S(y, n_1(x)), \quad I_4(x,y) = S(n_2(x), y).$$

The the following properties hold.

(1) (I_1, I_2) is a pair of SE-implications with

$$I_1(T_{21}(x,y), z) = I_1(x, I_1(y,z)),$$

$$I_2(T_{12}(x,y), z) = I_2(y, I_2(x,z)).$$

(2) If $x \le y$ iff $I_1(x,y) = \top$ iff $I_2(x,y) = \top$, then

$$\begin{aligned}
&T_{12}(x,y) \le z \text{ iff } y \le I_2(x,z)\\
&\quad \text{iff } x \le I_1(y,z) \text{ iff } T_{21}(x,y) \le z.
\end{aligned}$$

Moreover, $T_{12}(x,y) = T_{21}(x,y)$.

(3) (I_3, I_4) is a pair of SE-implications with

$$I_3(T_{21}(x,y), z) = I_3(x, I_3(y,z)),$$

$$I_4(T_{12}(x,y), z) = \mathcal{I}_4(y, I_4(x,z)).$$

(4) If $x \le y$ iff $I_3(x,y) = \top$ iff $I_4(x,y) = \top$, then

$$\begin{aligned}
&T_{12}(x,y) \le z \text{ iff } y \le I_4(x,z)\\
&\quad \text{iff } x \le I_3(y,z) \text{ iff } T_{21}(x,y) \le z.
\end{aligned}$$

Moreover, $T_{12}(x,y) = T_{21}(x,y)$.

(5) (I_1, I_3) *is a pair of E-implications such that*

$$I_1(T_{21}(x,y),z) = I_1(x, I_1(y,z)),$$

$$I_3(T_{21}(x,y),z) = I_3(x, I_3(y,z)),$$

$$I_1(I_3(x,\perp),\perp) = I_3(I_1(x,\perp),\perp) = n_1n_1(x).$$

(6) *If $x \le y$ iff $I_1(x,y) = \top$ iff $I_3(x,y) = \top$, then*

$$T_{21}(x,y) \le z \text{ iff } x \le I_1(y,z) \text{ iff } x \le I_3(y,z).$$

(7) (I_2, I_4) *is a pair of E-implications such that*

$$I_2(T_{12}(x,y),z) = I_2(y, I_2(x,z)),$$

$$I_4(T_{12}(x,y),z) = I_4(y, I_4(x,z)),$$

$$I_2(I_4(x,\perp),\perp) = I_4(I_2(x,\perp),\perp) = n_2n_2(x).$$

(8) *If $x \le y$ iff $I_2(x,y) = \top$ iff $I_4(x,y) = \top$, then*

$$T_{12}(x,y) \le z \text{ iff } y \le I_4(x,z) \text{ iff } y \le I_2(x,z).$$

(9) (I_1, I_4) *is a pair of S-implications such that*

$$I_1(T_{21}(x,y),z) = I_1(x, I_1(y,z)),$$

$$I_4(T_{12}(x,y),z) = I_4(y, I_4(x,z)).$$

(10) *If $S(n_1(x), S(n_2(y),z)) = S(n_2(y), S(n_1(x),z))$, then (I_1, I_4) is a pair of SE-implications.*

(11) (I_2, I_3) *is a pair of S-implications such that*

$$I_2(T_{12}(x,y),z) = I_2(y, I_2(x,z)),$$

$$I_3(T_{21}(x,y),z) = I_3(x, I_3(y,z)).$$

(12) *If $S(S(x, n_1(y)), n_2(z)) = S(S(x, n_2(z)), n_1(y))$, then (I_2, I_3) is a pair of SE-implications.*

Proof (1) (I1) $I_i(\top,\top) = I_i(\perp,\top) = I_i(\perp,\perp) = \top$ and $I_i(\top,\perp) = \perp$ for $i = \{1,2\}$.

(I2) If $x \le y$, then $n_i(x) \ge n_i(y)$. Hence $I_i(x,z) \ge I_i(y,z)$ for $i = \{1,2\}$.

(I3) $I_1(\top,x) = S(n_1(\top),x) = x$ and $I_2(\top,x) = S(x, n_2(\top)) = x$.

(E)

$$\begin{aligned}
&I_1(x, I_2(y,z)) = S(n_1(x), I_2(y,z)))\\
&= S(n_1(x), S(z, n_2(y))) = S(S(n_1(x),z), n_2(y))\\
&= S((I_1(x,z)), n_2(y)) = I_2(y, I_1(x,z)).
\end{aligned}$$

(S) $I_1(x,\perp) = S(n_1(x),\perp) = n_1(x)$ and $I_2(x,\perp) = S(\perp, n_2(x)) = n_2(x)$. Moreover, $I_2(I_1(x,\perp),\perp) = n_2(n_1(x)) = x$ and $I_1(I_2(x,\perp),\perp) = n_1(n_2(x)) = x$. Hence (I_1, I_2) is a pair of SE-implications. Moreover, we have

$$\begin{aligned}
&I_1(T_{21}(x,y),z) = S(n_1((T_{21}(x,y))),z)\\
&= S(n_1(n_2S(n_1(x),n_1(y)))),z)\\
&= S(S(n_1(x),n_1(y)),z) = S(n_1(x),S(n_1(y),z))\\
&= S(n_1(x),I_1(y,z)) = I_1(x,I_1(y,z)).
\end{aligned}$$

$$\begin{aligned}
&I_2(T_{12}(x,y),z) = S(z,n_2(T_{12}(x,y)))\\
&= S(z,n_2(n_1(S(n_2(x),n_2(y)))))\\
&= S(z,S(n_2(x),n_2(y))) = S(S(z,n_2(x)),n_2(y))\\
&= S(I_2(x,z),n_1(y)) = I_2(y,I_2(x,z)).
\end{aligned}$$

(2) Since $x \leq y$ iff $I_1(x,y) = \top$ iff $I_2(x,y) = \top$, by (1), then

$$\begin{aligned}
&I_2(T_{12}(x,y),z) = I_2(y,I_2(x,z)) = \top\\
&\text{iff } T_{12}(x,y) \leq z \text{ iff } y \leq I_2(x,z)\\
&\text{iff } \top = I_1(y,I_2(x,z)) = I_2(x,I_1(y,z))\\
&\text{iff } x \leq I_1(y,z) \ \text{ iff } \top = I_1(x,I_1(y,z)) = I_1(T_{21}(x,y),z)\\
&\text{iff } T_{21}(x,y) \leq z
\end{aligned}$$

Since $T_{12}(x,y) \leq z$ iff $T_{21}(x,y) \leq z$, then $T_{12}(x,y) = T_{21}(x,y)$.

(3) First, we show that (I_1, I_2) is a pair of SE-implications. We only show the conditions (E) and (S) because other cases are easily proved.

$$\begin{aligned}
&I_3(x,I_4(y,z)) = S(I_4(y,z)),n_1(x))\\
&= S(S(n_2(y),z),n_1(x)) = S(n_2(y),S(z,n_1(x)))\\
&= S(n_2(y),I_3(x,z)) = I_4(y,I_3(x,z)).
\end{aligned}$$

$I_3(x,\perp) = S(\perp,n_1(x)) = n_1(x)$ and $I_4(x,\perp) = S(n_2(x),\perp) = n_2(x)$. Moreover, $I_4(I_3(x,\perp),\perp) = n_2(n_1(x)) = x$ and $I_3(I_4(x,\perp),\perp) = n_1(n_2(x)) = x$. Second, we have

$$\begin{aligned}
&I_3(T_{21}(x,y),y) = S(z,n_1(T_{21}(x,y)))\\
&= S(x,n_1(n_2(S(n_1(x),n_1(y)))))\\
&= S(z,S(n_1(x),n_1(y))) = S(S(z,n_1(x)),n_1(y))\\
&= S(I_3(x,z),n_1(y)) = I_3(y,I_3(x,z)),
\end{aligned}$$

$$\begin{aligned}
&I_4(T_{12}(x,y),z) = S(n_2((T_{12}(x,y))),z)\\
&= S(n_2(n_1(S(n_2(x),n_2(y)))),z)\\
&= S(S(n_2(x),n_2(y)),z) = S(n_2(x),S(n_2(y),z))\\
&= S(n_2(x),I_4(y,z)) = I_4(x,I_4(y,z)).
\end{aligned}$$

(4) It is similarly proved as (2).

(5) We only show the condition (E) because other cases are easily proved.

$$\begin{aligned} I_1(x, I_3(y,z)) &= I_1(x, S(z, n_1(y))) \\ &= S(n_1(x), S(z, n_1(y))), \\ I_3(y, I_1(x,z)) &= I_3(x, S(n_1(y), z)) \\ &= S(S(n_1(x), z), n_1(y))). \end{aligned}$$

By (1) and (3),

$$I_1(T_{21}(x,y), z) = I_1(x, I_1(y,z)),$$

$$I_3(T_{21}(x,y), z) = I_3(x, I_5(y,z)).$$

(6) It is easily proved from (5) and (2).

(7) We only show the condition (E) because other cases are easily proved.

$$\begin{aligned} I_2(x, I_4(y,z)) &= I_2(x, S(n_2(y,z))) \\ &= S(S(n_2(y,z)), n_2(x)), \\ I_4(y, I_2(x,z)) &= I_4(y, S(z, n_2(x))) \\ &= S(n_2(y), S(z, n_2(x))). \end{aligned}$$

(8) It similarly proved as (6).

(9) It easily proved from (1) and (3).

(10) Since $S(n_1(x), S(n_2(y), z)) = S(n_2(y), S(n_1(x), z))$, then $I_1(x, I_4(y,z)) = I_4(y, I_1(x,z))$ from:

$$\begin{aligned} I_1(x, I_4(y,z)) &= I_1(x, S(n_2(y), z)) \\ &= S(n_1(x), S(n_2(y), z)), \\ I_4(y, I_1(x,z)) &= I_4(y, S(n_1(x), z)) \\ &= S(n_2(y), S(n_1(x), z)). \end{aligned}$$

(11) It easily proved from (5) and (7).

(12) Since $S(S(x, n_1(y)), n_2(z)) = S(S(x, n_2(z)), n_1(y))$, then $I_2(x, I_3(y,z)) = I_3(y, I_2(x,z))$ from:

$$\begin{aligned} I_2(x, I_3(y,z)) &= I_2(x, S(z, n_1(y))) \\ &= S(S(z, n_1(y)), n_2(x)), \\ I_3(y, I_2(x,z)) &= I_3(y, S(z, n_2(x))) \\ &= S(S(z, n_2(x)), n_1(y)). \end{aligned}$$

Example 3.4 Put $L = \{(x,y) \in R^2 \mid (0,1) \leq (x,y) \leq (2,3)\}$, S a pseudo t-conorm and (n_1, n_2) be a pair of negations in Example 3.2.

(1)

$$\begin{aligned} &I_1((x_1,y_1),(x_2,y_2)) = S(n_1(x_1,y_1),(x_2,y_2)) \\ &= S((2 - \tfrac{3x_1}{y_1}, \tfrac{3}{y_1}),(x_2,y_2)) = (x_2 + (2 - \tfrac{3x_1}{y_1})y_2, \tfrac{3y_2}{y_1}) \wedge (2,3). \end{aligned}$$

$$\begin{aligned}&I_2((x_1,y_1),(x_2,y_2)) = S((x_2,y_2),n_2(x_1,y_1))\\&= S((x_2,y_2),(\tfrac{2-x_1}{y_1},\tfrac{3}{y_1})) = (\tfrac{2-x_1+3x_2}{y_1},\tfrac{3y_2}{y_1})\wedge(2,3).\end{aligned}$$

$$\begin{aligned}&I_3((x_1,y_1),(x_2,y_2)) = S((x_2,y_2),n_1(x_1,y_1))\\&= S((x_2,y_2),(2-\tfrac{3x_1}{y_1},\tfrac{3}{y_1})) = (2+\tfrac{3(x_2-x_1)}{y_1},\tfrac{3y_2}{y_1})\wedge(2,3).\end{aligned}$$

$$\begin{aligned}&I_4((x_1,y_1),(x_2,y_2)) = S(n_2(x_1,y_1),(x_2,y_2))\\&= S((\tfrac{2-x_1}{y_1},\tfrac{3}{y_1}),(x_2,y_2)) = (x_2+(\tfrac{2-x_1}{y_1})y_2,\tfrac{3y_2}{y_1})\wedge(2,3).\end{aligned}$$

(2) The converse of Theorem 3.3(2) is not true for which $T_{12} = T_{21}$, but

$$I_1((\frac{1}{3},2),(0,2)) = (2,3) \text{ but} (\frac{1}{3},2) \not\leq (0,2),$$

$$I_2((2,2),(\frac{4}{3},2)) = (2,3) \text{ but} (2,2) \not\leq (\frac{4}{3},2).$$

(3) Since

$$\begin{aligned}&I_3((x_1,y_1),(x_2,y_2)) = (2,3)\\&\text{iff } (2+\tfrac{3(x_2-x_1)}{y_1},\tfrac{3y_2}{y_1}) \geq (2,3)\\&\text{iff } \tfrac{3y_2}{y_1} > 3 \text{ or } \tfrac{3y_2}{y_1} = 3, 2+\tfrac{3(x_2-x_1)}{y_1} \geq 2\\&\text{iff } y_1 < y_2 \text{ or } y_1 = y_2, x_1 \leq x_2\\&\text{iff } (x_1,y_1) \leq (x_2,y_2),\end{aligned}$$

$$\begin{aligned}&I_4((x_1,y_1),(x_2,y_2)) = (2,3)\\&\text{iff } (x_2+(\tfrac{2-x_1}{y_1})y_2,\tfrac{3y_2}{y_1}) \geq (2,3)\\&\text{iff } \tfrac{3y_2}{y_1} > 3 \text{ or } \tfrac{3y_2}{y_1} = 3, x_2+(\tfrac{2-x_1}{y_1})y_2 \geq 2\\&\text{iff } y_1 < y_2 \text{ or } y_1 = y_2, x_1 \leq x_2\\&\text{iff } (x_1,y_1) \leq (x_2,y_2),\end{aligned}$$

by Theorem 3.3 (6), we have

$$\begin{aligned}&T_{12}((x_1,y_1),(x_2,y_2)) \leq (x_3,y_3)\\&\text{iff } (x_2,y_2) \leq I_4((x_1,y_1),(x_3,y_3))\\&\text{iff } (x_1,y_1) \leq I_3((x_2,y_2),(x_3,y_3))\\&\text{iff } T_{21}((x_1,y_1),(x_2,y_2)) \leq (x_3,y_3).\end{aligned}$$

Moreover, $T_{12}((x_1,y_1),(x_2,y_2)) = T_{21}((x_1,y_1),(x_2,y_2))$.

(4) (I_1, I_3) is not a pair of S-implications from:

$$\begin{aligned}&I_1(I_3((2,2),(0,1)),(0,1)) = n_1(n_1(2,2))\\&= (4,2) = I_3(I_1((2,2),(0,1)),(0,1)) \neq (2,2).\end{aligned}$$

(5) (I_2, I_4) is not a pair of S-implications from:

$$\begin{aligned}&I_2(I_4((2,2),(0,1)),(0,1)) = n_2(n_2(2,2))\\&= (\tfrac{4}{3},2) = I_4(I_2((2,2),(0,1)),(0,1)) \neq (2,2).\end{aligned}$$

(6) (I_1, I_4) is not a pair of E-implications from:

$$I_1((-1,\tfrac{3}{2}), I_4((3,1),(1,1))) = S(n_1(-1,\tfrac{3}{2}), S(n_2(3,1),(1,1)))$$
$$= S((4,2), S((\tfrac{2}{3},1),(1,1))) = (\tfrac{17}{3},2),$$

$$I_4((3,1), I_1((-1,\tfrac{3}{2}),(1,1))) = S(n_2((3,1), S(n_1(-1,\tfrac{3}{2}),(1,1)))$$
$$= S((\tfrac{2}{3},1), S((4,2),(1,1))) = (\tfrac{19}{3},2).$$

(7) (I_2, I_3) is not a pair of E-implications from:

$$I_2((0,3), I_3((-1,\tfrac{3}{2}),(1,1))) = S(S((1,1), n_1(-1,\tfrac{3}{2})), n_2(0,3))$$
$$= S(S((1,1),(4,2)),(\tfrac{2}{3},1)) = (\tfrac{20}{3},2),$$

$$I_3((-1,\tfrac{3}{2}), I_2((0,3),(1,1))) = S(S((1,1), n_2(0,3)), n_1(-1,\tfrac{3}{2}))$$
$$= S(S((1,1),(\tfrac{2}{3},1)),(4,2)) = (\tfrac{22}{3},2).$$

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