Mathematica Aeterna, Vol. 3, 2013, no. 8, 655 - 665

A time periodic solution for the viscous Cahn-Hilliard equation with varying mobility

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Abstract

In this paper, we study the viscous Cahn-Hilliard equation with varying mobility and time periodic coefficients. We use the energy method to get some priori estimates and prove the existence of solutions by using the Leray-Schauder fixed point theorem.

Mathematics Subject Classification: 35L25, 35B10, 35A05, 47H10.
Keywords: viscous Cahn-Hilliard equation, varying mobility, periodic solution, Leray-Schauder's fixed point theorem.

1 Introduction

We consider the viscous Cahn-Hilliard equation with a varying mobility in one spatial dimension

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} + D[m(u)(\gamma D^3 u - D\varphi(u,t))] = f(x,t), \quad (x,t) \in Q_T, \quad (1.1)$$

where $Q_T \equiv (0,1) \times (0,T), D = \frac{\partial}{\partial x}, k > 0$ is a viscous coefficient, $\gamma > 0$ is a constant, and $m(u) \in C^{1+\alpha}(R)$, $\alpha \in (0,1)$, is a mobility. $\varphi(u,t) = a(t)u^3 - b(t)u$ represents a potential, $a(t), b(t) \in C^{1+\alpha}(R)$ are positive periodic functions with period $T > 0$, and $f(x,t) \in C^{2+\alpha,1+\frac{\alpha}{4}}(\overline{Q_T})$ stands for a source with $f(x,t) = f(x,t+T)$ for all $(x,t) \in Q_T$.

We supplement (1.1) with the natural boundary conditions

$$Du(0,t) = Du(1,t) = D^3u(0,t) = D^3u(1,t) = 0, \quad t \in R^+ \tag{1.2}$$

and the time-periodic condition

$$u(x,t) = u(x,t+T), \quad x \in (0,1). \tag{1.3}$$

In addition, we assume that u satisfies the following equation:

$$\int_0^1 u(x,t)dx = 0. \tag{1.4}$$

The viscous Cahn-Hilliard equation is often used to account for viscosity effects in the phase of separation of the polymer-polymer systems, see [1], and Murray [6] also proposed a similar model to describe the diffusion phenomenon of species. In recent years, there have been numerous investigations on the viscous Cahn-Hilliard equation with a constant mobility, for example, the existence and uniqueness of classical solutions, the asymptotic estimates and asymptotic behavior of solutions, the large time behavior of solutions, and so on (cf. [2,3]). Equation (1.1) with varying mobility, viscosity and source terms more exactly reflects the physical reality to a certain extent. However, as far as we know, there are only few investigations on the viscous Cahn-Hilliard equation with a varying mobility. Yin and Liu [8] considered the existence and uniqueness of classical solutions and the large time behavior of solutions.

Recently, many scholars have paid much attention to the time-periodic solutions of the Cahn-Hilliard type equations with a constant mobility, (cf. [3,9,10]). Accessing the relevant papers, we find that the research on time periodic solutions of the viscous Cahn-Hilliard equation with a varying mobility has not been started yet. Motivated by this observation, we consider the existence of time periodic solutions.

Here, inspired with the ideas of Murray [6], we give a sketch of the derivation of equation (1.1) by modeling the cell growth. Let $u(x,t)$ be the cell concentration at the position x and time t. By the conservation law, we know that

$$\frac{\partial u}{\partial t} = \operatorname{div} \overrightarrow{J} + f(x,t),$$

where $\overrightarrow{J}$ represents the diffusion flow and $f(x,t)$ is a source. In general, the flux depends on concentration and gradient; that is, $\overrightarrow{J} = -m(u)\Phi(u, \nabla u)$. It is easy to see that the diffusion phenomena described by $\overrightarrow{J}$ is local or within a small range, and this form is suitable for thin systems only, like the situation of small concentration, see [6]. However, in the special situation, like the embryonic development, the cell concentration is relatively high. The classic model that is fixed in thin system is not very precise. Therefore, we need one model to describe the non-partial or the wide large-scale diffusion. We no longer only consider the situation that $\overrightarrow{J} \propto -m(u)\Phi(u, \nabla u)$. Hence, we use $< u >$ instead of u, where $< u >$ indicates that u may affect the average density in the region that u takes value at position x. Especially, considering homogeneous regions, we take the ball $B(x, R)$ as an example, namely

$$< u(x,t) > \equiv \frac{1}{\omega_n} \int_{B_R} u(x+r,t)dr,$$

where ω_n represents the volume of the ball with radius R in R^n. When r is very small, expanding $u(x+r,t)$ by using a Taylor expansion with respect to x and substituting it into the right hand side of the above integral, neglecting the second-order and higher order terms, and unifying the symmetry, the above integral becomes

$$\begin{aligned} < u(x,t) > &\equiv \tfrac{1}{\omega_n}[u(x,t) \textstyle\int_{B_R(x)} dr + \nabla^2 u(x,t) \int_{B_R(x)} \frac{r^2}{2} dr] \\ &= u(x,t) + \tfrac{nR^2}{2(n+2)} \nabla^2 u(x,t). \end{aligned}$$

As described in [6], for simplicity, we select the flux as follows:

$$\overrightarrow{J} = -m(u)(\Phi(u, \nabla u) - \gamma \nabla(\nabla^2 u)).$$

Here, we are interested in considering the case that $\Phi(u, \nabla u) = (3au^2 - b)\nabla u$. By the conservation equation, we get

$$\frac{\partial u}{\partial t} = -\text{div}(m(u)(\gamma D^3 u - (3au^2 - b)Du)) + f(x,t). \tag{1.4}$$

In particular, if the cell concentration is sensitive to seasons, it is reasonable to investigate the existence of time-periodic solution of equation (1.4). Therefore, we consider the coefficients a, b, and the function f as the time-periodic ones, and assuming the growth of cells is affected by viscosity, we obtain equation (1.1) immediately.

The rest of this paper is organized as follows: In Section 2, we consider the priori estimates and Hölder norm estimates of solutions. We study the existence of time-periodic solutions for problem (1.1)-(1.3) by the Leray-Schauder fixed point theorem in Section 3. Here, we need the following conditions:

H1. There exist positive constants $\overline{a}, \underline{a}$, and $\overline{b}$ such that $0 < \underline{a} \le a(t) < \overline{a}$ and $0 < b(t) < \overline{b}$;
H2. There exist positive constants C_0 and C_1 such that $|Dm(u)| \le C_0$ and $0 < m(u) < \frac{C_1}{1+u^2}$.

2 Preliminary results

In this section, we obtain some priori estimates and Hölder norm estimates of solutions.

Lemma 2.1 *If the assumptions (H1)-(H2) hold and* u *is a solution of (1.1)-(1.3), then*

$$\|u\|_\infty \le C, \tag{2.1}$$

$$\|Du\|_\infty \le C, \tag{2.2}$$

$$\int\int_{Q_T} m(u)|D^3u(x,t)|^2 dxdt \le C, \tag{2.3}$$

where C *is a generic positive constant not necessarily the same.*

Proof: Let C denote a generic positive constant that is not necessarily the same at different occurrences. Multiplying both sides of (1.1) by u and integrating the result over Q_T, we obtain

$$\begin{gathered}\gamma \int\int_{Q_T} m(u)|D^2u|^2 dxdt = -\gamma \int\int_{Q_T} Dm(u)D^2uDudxdt \\ -\int\int_{Q_T} m(u)(3a(t)u^2 - b(t))|Du|^2 dxdt + \int\int_{Q_T} fudxdt \\ \le C\int\int_{Q_T} |D^2uDu|dxdt + C\int\int_{Q_T} |Du|^2 dxdt + C\int\int_{Q_T} u^2 dxdt + C \\ \le \frac{\gamma}{2}\int\int_{Q_T} m(u)|D^2u|^2 dxdt + C\int\int_{Q_T} u^2 dxdt + C.\end{gathered}$$

We then have

$$\int\int_{Q_T} |D^2u|^2 dxdt \le C\int\int_{Q_T} u^2 dxdt + C, \tag{2.4}$$

$$\int\int_{Q_T} m(u)u^2|Du|^2 dxdt \le C\int\int_{Q_T} |D^2uDu|dxdt + C\int\int_{Q_T} |Du|^2 dxdt + \int\int_{Q_T} fudxdt$$

$$\le C\int\int_{Q_T} |D^2u|^2 dxdt + C\int\int_{Q_T} u^2 dxdt + C \le C\int\int_{Q_T} u^2 dxdt + C. \tag{2.5}$$

By (2.4) and (2.5), we get

$$\int\int_{Q_T} |Du|^2 dxdt = -\int\int_{Q_T} u\cdot D^2udxdt \le C\int\int_{Q_T} |D^2u|^2 dxdt + C\int\int_{Q_T} u^2 dxdt$$

$$\le C\int\int_{Q_T} u^2 dxdt + C. \tag{2.6}$$

It follows from (1.4) that for any given $t \in R^+$ there exists a point $x_t \in (0,1)$ such that $u(x_t,t) = 0$. Thus, we have

$$|u^4(x,t)| = |u^4(x,t) - u^4(x_t,t)| = |\int_{x_t}^x D(u^4(y,t))dy| \le \int_0^1 4u^2|uDu|dx.$$

Integrating the above inequality over Q_T and using Young's inequality, we have

$$\int\int_{Q_T} u^4 dxdt \leq 4\int\int_{Q_T} u^2|uDu|dxdt \leq \frac{1}{2}\int\int_{Q_T} u^4 dxdt + 8\int\int_{Q_T} |uDu|^2 dxdt.$$

Combining the above inequality with (2.5), we get

$$\int\int_{Q_T} u^4 dxdt \leq 16\int\int_{Q_T} |uDu|dxdt \leq M\int\int_{Q_T} u^2 dxdt + C. \tag{2.7}$$

Using Young's inequality again, we obtain

$$\int\int_{Q_T} u^2 dxdt \leq \frac{1}{2M}\int\int_{Q_T} u^4 dxdt + C \leq \frac{1}{2}\int\int_{Q_T} u^2 dxdt + C,$$

that is,

$$\int\int_{Q_T} u^2 dxdt \leq C. \tag{2.8}$$

Substituting (2.8) into (2.4), (2.5), (2.6), and (2.7), we obtain

$$\int\int_{Q_T} |D^2u|^2 dxdt \leq C, \tag{2.9}$$

$$\int\int_{Q_T} u^2|Du|^2 dxdt \leq C, \tag{2.10}$$

$$\int\int_{Q_T} |Du|^2 dxdt \leq C, \tag{2.11}$$

$$\int\int_{Q_T} u^4 dxdt \leq C. \tag{2.12}$$

Consequently, we have the inequality

$$\int\int_{Q_T} (|Du|^2 + k|D^2u|^2) dxdt \leq C. \tag{2.13}$$

from inequalities (2.9)–(2.12).

Multiplying both sides of (1.1) by D^2u and integrating the result with respect to x over (0,1), using hypotheses (H1)-(H2), and Young's inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\int_0^1 |Du|^2 dx + k\int_0^1 |D^2u|^2 dx\right) + 2\gamma\int_0^1 m(u)|D^3u|^2 dx \\
&= 2\int_0^1 m(u)(3a(t)u^2 - b(t))DuD^3u dx + 2\int_0^1 f(x,t)D^2u dx \\
&\leq C\int_0^1 |DuD^3u| dx + C\int_0^1 |D^2u|^2 dx + C
\end{aligned}$$

$$\le \gamma\int_0^1 m(u)|D^3u|^2dx + C\int_0^1 |Du|^2dx + C\int_0^1 |D^2u|^2dx + C.$$

Consequently, we get

$$\frac{d}{dt}(\int_0^1 |Du|^2dx + k\int_0^1 |D^2u|^2dx) + \gamma\int_0^1 m(u)|D^3u|^2dx$$
$$\le C\int_0^1 |Du|^2dx + C\int_0^1 |D^2u|^2dx + C. \tag{2.14}$$

We then have

$$\frac{d}{dt}(\int_0^1 |Du|^2dx + k\int_0^1 |D^2u|^2dx)$$
$$\le C[\int_0^1 |Du|^2dx + k\int_0^1 |D^2u|^2dx] + C. \tag{2.15}$$

From (2.13), we know that there exists a point $\widetilde{t} \in (0,T)$ such that

$$\int_0^1 (|Du(x,\widetilde{t})|^2 + k|D^2u(x,\widetilde{t})|^2)dx = \frac{1}{T}\int\int_{Q_T} (|Du|^2 + k|D^2u|^2)dxdt \le C.$$

Then, for any fixed point $t \in (\widetilde{t}, T]$, integrating (2.15) from $\widetilde{t}$ to t, we have

$$\int_0^1 ((Du(x,t))^2 + k(D^2u(x,t))^2)dx$$

$$\le \int_0^1 (|Du(x,\widetilde{t})|^2 + k|D^2u(x,\widetilde{t})|^2)dx + C\int\int_{Q_T} (|Du|^2 + k|D^2u|^2)dxdt + C \le C.$$

Recalling the periodicity of u, we get

$$\int_0^1 ((Du(x,0))^2 + k(D^2u(x,0))^2)dx = \int_0^1 ((Du(x,T))^2 + k(D^2u(x,T))^2)dx \le C.$$

Then, for any fixed point $t \in [0,T]$, we have

$$\int_0^1 ((Du(x,t))^2 + k(D^2u(x,t))^2)dx \le C.$$

We then have

$$\int_0^1 (Du(x,t))^2dx \le C, \tag{2.16}$$

$$\int_0^1 (D^2u(x,t))^2dx \le C. \tag{2.17}$$

Integrating (2.14) from 0 to T, we get

$$\int\int_{Q_T} m(u)|D^3u(x,t)|^2dxdt \le C.$$

It follows from (1.5) that for any given $t \in R^+$ there exists a point $x_t \in (0,1)$ such that $u(x_t, t) = 0$. Thus,

$$|u(x,t)| = |u(x,t) - u(x_t,t)| = |\int_{x_t}^{x} D(u(y,t))dy| \le (\int_0^1 (Du(x,t))^2 dx)^{\frac{1}{2}} \le C,$$

i.e.,

$$\|u\|_\infty \le C.$$

Combining (2.16) with (2.17) yields

$$\|Du\|_\infty \le C.$$

The proof is completed.

As an important step, we give the Hölder norm estimate on the local solution in time.

Lemma 2.2 *Suppose that (H1)–(H2) hold and u is a solution of (1.1)-(1.3). Then there exist two different constants C depending only on the known quantities, such that for any* $(x_1,t_1),(x_2,t_2) \in Q_T$,

$$|u(x_1,t_1) - u(x_2,t_2)| \le C(|x_1 - x_2|^{\frac{1}{2}} + |t_1 - t_2|^{\frac{1}{8}}), \tag{2.18}$$

$$|Du(x_1,t_1) - Du(x_2,t_2)| \le C(|x_1 - x_2|^{\frac{1}{2}} + |t_1 - t_2|^{\frac{1}{8}}). \tag{2.19}$$

Proof: Obviously, inequality (2.18) is equivalent to

$$|u(x_1,t) - u(x_2,t)| \le C|x_1 - x_2|^{\frac{1}{2}} \tag{2.20}$$

and

$$|u(x,t_1) - u(x,t_2)| \le C|t_1 - t_2|^{\frac{1}{8}}. \tag{2.21}$$

In fact, (2.20) can be directly obtained by (2.16). To prove (2.21), we only need to consider the case that $0 \le x \le \frac{1}{2}, \Delta t = t_2 - t_1 > 0$, and $(\Delta t)^{\frac{1}{4}} \le \frac{1}{4}$. Set $L = (I - kD^2)^{-1}$ and $Lg = \omega$, namely, ω satisfies

$$(I - kD^2)\omega = g, \quad x \in \Omega,$$

$$n \cdot \Delta\omega = 0, \quad x \in \partial\Omega.$$

It is easily seen that $\int_\Omega |\omega|^2 dx \le \int_\Omega |g|^2 dx$, i.e.,

$$\int_\Omega |Lg|^2 dx \le \int_\Omega |g|^2 dx.$$

Now, we can rewrite equation (1.1) as

$$(I - kD^2)\frac{\partial u}{\partial t} + D[m(u)(\gamma D^3 u - D\varphi(u,t))] = f(x,t).$$

Using the operator L for the above equation, we have

$$\frac{\partial u}{\partial t} + LD[m(u)(\gamma D^3 u - D\varphi(u,t))] = Lf(x,t). \tag{2.22}$$

Integrating equation (2.22) with respect to x over $(y, y+(\Delta t)^{\frac{1}{4}}) \times (t_1, t_2)$, where $0 < t_1 < t_2 < T, \Delta t = t_2 - t_1$, and noting $y' = y + (\Delta t)^{\frac{1}{4}}$, one can see that

$$\int_y^{y+(\Delta t)^{\frac{1}{4}}} [u(z,t_2) - u(z,t_1)]dz = \int_{t_1}^{t_2} L[m(u(y',s))(\gamma D^3 u(y',s) - D\varphi(u(y',s),s))$$
$$-m(u(y,s))(\gamma D^3 u(y,s) - D\varphi(u(y,s),s))]ds + \int_{t_1}^{t_2} \int_y^{y+(\Delta t)^{\frac{1}{4}}} Lf(z,s)dzds. \tag{2.23}$$

Setting

$$N(y,s) = L[m(u(y',s))(\gamma D^3 u(y',s) - D\varphi(u(y',s),s))$$
$$-m(u(y,s))(\gamma D^3 u(y,s) - D\varphi(u(y,s),s))] - \int_y^{y+(\Delta t)^{\frac{1}{4}}} Lf(z,s)dz.$$

equation (2.23) can be converted into

$$(\Delta t)^{\frac{1}{4}} \int_0^1 [u(y + \theta(\Delta t)^{\frac{1}{4}}, t_2) - u(y + \theta(\Delta t)^{\frac{1}{4}}, t_1)]d\theta = -\int_{t_1}^{t_2} N(y,s)ds. \tag{2.24}$$

Integrating both sides of (2.24) with respect to y over $(x, x + (\Delta t)^{\frac{1}{4}})$, we get

$$(\Delta t)^{\frac{1}{2}}(u(x^*, t_2) - u(x^*, t_1)) = -\int_x^{x+(\Delta t)^{\frac{1}{4}}} \int_{t_1}^{t_2} N(y,s)dsdy,$$

where $x^* = y^* + \theta^*(\Delta t)^{\frac{1}{4}}, \theta^* \in (0,1)$, and $y^* \in (x, x + (\Delta t)^{\frac{1}{4}})$. Here, we have used the mean value theorem. Hence, by Hölder's inequality, and (2.1) and (2.3), we get

$$|u(x^*, t_2) - u(x^*, t_1)|^2 \Delta t \le C(\Delta t)^{\frac{5}{4}},$$

that is,

$$|u(x^*, t_2) - u(x^*, t_1)| \le C(\Delta t)^{\frac{1}{8}}.$$

Combining the above inequality with (2.20), we have

$$|u(x,t_2) - u(x,t_1)| \le |u(x,t_2) - u(x^*,t_2)| + |u(x^*,t_2) - u(x^*,t_1)| + |u(x,t_1) - u(x^*,t_1)|$$
$$\le C|t_1 - t_2|^{\frac{1}{8}}.$$

Inequality (2.19) is equivalent to

$$|Du(x_1,t) - Du(x_2,t)| \le C|x_1 - x_2|^{\frac{1}{2}} \tag{2.25}$$

and

$$|Du(x,t_1) - Du(x,t_2)| \le C|t_1 - t_2|^{\frac{1}{8}}. \tag{2.26}$$

Integrating both sides of (1.1) over $(0,x) \times (t_1,t_2)$ yields the equation

$$Du(x,t_2) - Du(x,t_1) = \int_0^x [u(z,t_2) - u(z,t_1)]dz + \int_{t_1}^{t_2} m(u(x,s))\gamma D^3 u(x,s)ds$$
$$- \int_{t_1}^{t_2} m(u(x,s))\varphi'(u(x,s),s)Du(x,s)ds + \int_0^x \int_{t_1}^{t_2} f(z,s)dsdz.$$

Again, integrating the above equation with respect to x over $(y, y+(\Delta t)^{\frac{3}{4}})$, and using the mean value theorem and (2.21), we have

$$(\Delta t)^{\frac{3}{4}}|Du(x^*,t_2) - Du(x^*,t_1)| \le C(\Delta t)^{\frac{7}{8}},$$

that is,

$$|Du(x^*,t_2) - Du(x^*,t_1)| \le C(\Delta t)^{\frac{1}{8}}. \tag{2.27}$$

By (2.17), we obtain

$$|Du(x_1,t) - Du(x_2,t)| \le C|x_1 - x_2|^{\frac{1}{2}}. \tag{2.28}$$

Combining (2.27) with (2.28), we have

$$|Du(x,t_2) - Du(x,t_1)| \le C|t_2 - t_1|^{\frac{1}{8}}.$$

The proof of this lemma is completed.

3 The existence of time periodic solutions

In this section, we use Lemma 2.2 and Leray-Schauder's fixed point theorem to study the existence of time-periodic solutions to problem (1.1)-(1.3).

Theorem 3.1 *The problem (1.1)-(1.3) admits a time-periodic solution on Q_T under the hypotheses (H1)-(H2).*

Proof: Set

$$X = \{u : u \in C^{1+\alpha, 1+\frac{\alpha}{4}}(\overline{Q_T}), Du(0,t) = Du(1,t) = 0\}.$$

Obviously X is a Banach space. Define an operator on X as follows:

$$T : X \times R \to X, \quad (u,\sigma) \to v,$$

where v is a solution of the following linear problem:

$$\frac{\partial v}{\partial t} - k\frac{\partial D^2 v}{\partial t} + \gamma m(u)D^4 v + \gamma m'(u)DuD^3 v - \sigma m(u)\varphi'(u,t)D^2 v$$

$$-\sigma(m'(u)\varphi'(u,t)Du + m(u)\varphi''(u,t)Du)Dv = \sigma f(x,t), \qquad (x,t) \in Q_T,$$
$$Dv(0,t) = Dv(1,t) = D^3v(0,t) = D^3v(1,t) = 0, \qquad t \in [0,T],$$
$$v(x,t) = v(x,t+T), \quad x \in (0,1),$$

where $\sigma \in [0,1]$ is a parameter. From Lemma 2.2 we know that

$$p_1(x,t) = \gamma m(u), p_2(x,t) = \gamma m'(u)Du, p_3(x,t) = m(u)\varphi'(u,t),$$
$$p_4(x,t) = m'(u)\varphi'(u,t)Du + m(u)\varphi''(u,t)Du$$

are Hölder continuous, and hence, by the classical linear theory (cf. [4,5,7]), the above problem admits a unique solution $v \in C^{4+\beta,1+\frac{\beta}{4}}(\overline{Q_T})$. So, the operator T is well-defined and compact. When $\sigma = 0$, we have $T(u,0) = 0$. Moreover, if $v = T(u,\sigma)$ for some $\sigma \in (0,1]$, then u satisfies (1.1)–(1.2) and $u(x,0) = u(x,T)$. Thus, from the discussions above, one can see that $\|u\|_{C^{4+\beta,1+\frac{\beta}{4}}} \le A$ with A being a constant independent of u and σ. By the Leray-Schauder fixed point theorem, $T(u,1)$ admits a fixed point u, which is the desired periodic solution of problem (1.1)-(1.3).

Acknowledgments

This work is supported by the Natural Science Foundation of Shandong Province of China (ZR2012AM018) and the Fundamental Research Funds for the Central Universities (No.201362032). The authors would like to deeply thank all the reviewers for their insightful and constructive comments.

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Received: September, 2013