

# Global Behavior of a Higher Order Nonlinear Difference Equation

Guomei Tang, Hua Liu

School of Mathematics and Computer Science  
Northwest University for Nationalities  
Lanzhou, Gansu 730030, People's Republic of China

## Abstract

In this paper, we consider the higher order nonlinear rational difference equation

$$x_{n+1} = \frac{\alpha + x_n + \gamma x_{n-k}}{A + x_{n-k}}, \quad n = 0, 1, \dots$$

with the parameters and the initial conditions  $x_{-k}, \dots, x_0$  are nonnegative real numbers. We investigate the periodic character, invariant intervals and the global asymptotic stability of all positive solutions of the above mentioned equation. In particular, our results partially confirm the conjecture introduced by Amleh and Ladas in their paper.

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**Keywords:** Difference equation; Global asymptotic stability; Invariant intervals; Period-two solutions

## 1 Introduction and preliminaries

Our aim in this paper is to investigate the global behavior of solutions of the following nonlinear rational difference equation

$$x_{n+1} = \frac{\alpha + x_n + \gamma x_{n-k}}{A + x_{n-k}}, \quad n = 0, 1, \dots \quad (1)$$

where the parameters  $\alpha, \gamma, A$  and the initial conditions  $x_{-k}, \dots, x_0$  are nonnegative real numbers,  $k \in \{1, 2, \dots\}$ .

In 2007, Amleh and Ladas [1] proposed the following conjecture:

**Conjecture 1.1** Assume that  $\gamma, A \in [0, \infty)$  and  $\alpha, \gamma + A \in (0, \infty)$ . Show that the positive equilibrium of the following equation:

$$x_{n+1} = \frac{\alpha + x_n + \gamma x_{n-1}}{A + x_{n-1}}, \quad n = 0, 1, \dots \quad (2)$$

is globally asymptotically stable.

Inspired by the above conjecture, we will consider and investigate the global asymptotic stability and the invariant interval for all positive solutions of Eq.(1).

For the global behavior of solutions of some related equations, see [2,4,6,7]. Other related results can be found in [3,9–12]. For the sake of convenience, we recall some theorems which will be useful in the sequel.

**Theorem 1.2** (See [8]) Assume that  $P, Q \in R$  and  $k \in \{1, 2, \dots\}$ . Then

$$|P| + |Q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} = Py_n + Qy_{n-k}, \quad n = 0, 1, \dots$$

**Lemma 1.3** (See [5]) Consider the difference equation

$$y_{n+1} = f(y_n, y_{n-k}), \quad n = 0, 1, \dots \quad (3)$$

where  $k \in \{1, 2, \dots\}$ . Let  $I = [a, b]$  be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \rightarrow [a, b]$$

is a continuous function satisfying the following properties:

- (a)  $f(u, v)$  is nondecreasing in  $u$  and nonincreasing in  $v$ .
- (b) If  $(m, M) \in [a, b] \times [a, b]$  is a solution of the system

$$m = f(m, M) \quad \text{and} \quad M = f(M, m),$$

then  $m = M$ .

Then Eq.(3) has a unique equilibrium  $\bar{y} \in [a, b]$  and every solution of Eq.(3) converges to  $\bar{y}$ .

**Lemma 1.4** (See [9]) Let  $I = [a, b]$  be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \rightarrow [a, b]$$

is a continuous function satisfying the following properties:

- (a)  $f(u, v)$  is nondecreasing in each of its arguments.
- (b) The equation  $f(u, u) = u$  has a unique positive solution in the interval  $[a, b]$ .

Then Eq.(3) has a unique equilibrium  $\bar{y} \in [a, b]$  and every solution of Eq.(3) converges to  $\bar{y}$ .

## 2 Local stability and period-two solutions

Eq.(1) possesses the unique positive equilibrium

$$\bar{x} = \frac{1 + \gamma - A + \sqrt{(1 + \gamma - A)^2 + 4\alpha}}{2}.$$

The linearized equation associated with Eq.(1) about the positive equilibrium is

$$z_{n+1} - \frac{1}{A + \bar{x}}z_n - \frac{\gamma - \bar{x}}{A + \bar{x}}z_{n-k} = 0. \quad (4)$$

By Theorem 1.2, we have the following result.

**Theorem 2.1** Assume that

$$A > 1. \quad (5)$$

Then the positive equilibrium  $\bar{x}$  of Eq.(1) is locally asymptotically stable.

**Theorem 2.2** Eq.(1) has no nonnegative prime period-two solution.

**Proof.** (a) Assume that  $k$  is odd, then  $x_{n+1} = x_{n-k}$ . Let

$$\cdots, \phi_1, \phi_2, \phi_1, \phi_2, \cdots$$

be a nonnegative prime period-two solution of Eq.(1). Then  $\phi_1, \phi_2$  satisfy the following system:

$$\phi_1 = \frac{\alpha + \phi_2 + \gamma\phi_1}{A + \phi_1} \quad \text{and} \quad \phi_2 = \frac{\alpha + \phi_1 + \gamma\phi_2}{A + \phi_2}.$$

Substituting the above two equations, we obtain

$$(\phi_1 - \phi_2)(\phi_1 + \phi_2 - \gamma + A + 1) = 0.$$

If  $\phi_1 \neq \phi_2$ , then  $\phi_1 + \phi_2 = \gamma - A - 1$ .

Adding them and using the above equations, we can get

$$\phi_1\phi_2 = A + 1 - \gamma - \alpha.$$

Obviously, if  $\phi_1 + \phi_2 = \gamma - A - 1 \geq 0$ , then  $\phi_1\phi_2 = A + 1 - \gamma - \alpha < 0$ . This contradicts the hypothesis that  $\phi_1, \phi_2$  are nonnegative.

(b) Assume that  $k$  is even, then  $x_n = x_{n-k}$ . If there exist distinctive non-negative real number  $\phi_1$  and  $\phi_2$ , such that

$$\cdots, \phi_1, \phi_2, \phi_1, \phi_2, \cdots$$

is a prime period-two solution of Eq.(1) and  $\phi_1, \phi_2$  satisfy the following system:

$$\phi_1 = \frac{\alpha + \phi_2 + \gamma\phi_2}{A + \phi_2} \quad \text{and} \quad \phi_2 = \frac{\alpha + \phi_1 + \gamma\phi_1}{A + \phi_1},$$

which is equivalent to

$$A\phi_1 + \phi_1\phi_2 = \alpha + \phi_2 + \gamma\phi_2 \quad \text{and} \quad A\phi_2 + \phi_1\phi_2 = \alpha + \phi_1 + \gamma\phi_1.$$

Subtracting these two equation, we can get

$$(\phi_1 - \phi_2)(A + \gamma + 1) = 0.$$

Then  $\phi_1 = \phi_2$ . This contradicts the hypothesis that  $\phi_1 \neq \phi_2$ .

The proof is complete. □

### 3 Invariant Interval

In this section, we will investigate the invariant interval of Eq.(1).

Let

$$f(u, v) = \frac{\alpha + u + \gamma v}{A + v}.$$

Then the following statements are true.

**Lemma 3.1 (a)** *Assume that  $A\gamma - \alpha \leq 0$ . Then  $f(u, v)$  is increasing in  $u$  for each  $v$  and decreasing in  $v$  for each  $u$ .*

**(b)** *Assume that  $A\gamma - \alpha > 0$ . Then  $f(u, v)$  is increasing in  $u$  for each  $v$  and increasing in  $v$  for  $u \in [0, A\gamma - \alpha]$ , and decreasing in  $v$  for  $u \in [A\gamma - \alpha, \infty)$ .*

**Proof.** The proofs of (a) and (b) are simple and will be omitted. □

**Lemma 3.2** *Assume that (5) holds, then Eq.(1) possesses the following invariant intervals:*

**(a)**  $[\gamma, \frac{\alpha}{A-1}]$ , when  $A\gamma - \alpha < \gamma$ ;

**(b)**  $[\frac{\alpha}{A-1}, \gamma]$ , when  $A\gamma - \alpha \geq \gamma$ .

**Proof.**

(a) Using the monotonic character of the function  $f(u, v)$  which is described by Lemma 3.1 and the condition that  $A\gamma - \alpha < \gamma$ , when  $x_{-k}, \dots, x_{-1}, x_0 \in [\gamma, \frac{\alpha}{A-1}]$ , we can get

$$\gamma < f(\gamma, \frac{\alpha}{A-1}) \leq x_1 = \frac{\alpha + x_0 + \gamma x_{-k}}{A + x_{-k}} = f(x_0, x_{-k}) \leq f(\frac{\alpha}{A-1}, \gamma) < \frac{\alpha}{A-1}.$$

The proof follows by induction.

(b) Using the monotonic character of the function  $f(u, v)$  which is described by Lemma 3.1 (b) and the condition that  $A\gamma - \alpha \geq \gamma$ , we see that when  $x_{-k}, \dots, x_{-1}, x_0 \in [\frac{\alpha}{A-1}, \gamma]$ , then

$$\frac{\alpha}{A-1} \leq f(\frac{\alpha}{A-1}, \frac{\alpha}{A-1}) \leq x_1 = \frac{\alpha + x_0 + \gamma x_{-k}}{A + x_{-k}} \leq \frac{\alpha + \gamma + \gamma^2}{A + \gamma} \leq \gamma.$$

The proof follows by induction.

The proof is complete.  $\square$

## 4 Semicycles analysis

Let  $\{x_n\}_{n=-k}^\infty$  be a positive solution of Eq.(1). Then we have the following equations:

$$x_{n+1} - \frac{\alpha}{A} = \frac{x_n + (\gamma - \frac{\alpha}{A})x_{n-k}}{A + x_{n-k}}, \text{ for } n \geq 0. \quad (6)$$

$$x_{n+1} - \gamma = -\frac{(A\gamma - \alpha) - x_n}{A + x_{n-k}}, \text{ for } n \geq 0. \quad (7)$$

$$x_{n+1} - (A\gamma - \alpha) = \frac{A[\frac{\alpha}{A} - (A\gamma - \alpha)] + x_n + [\gamma - (A\gamma - \alpha)]x_{n-k}}{A + x_{n-k}}, \text{ for } n \geq 0. \quad (8)$$

$$x_{n+1} - \frac{\alpha}{A-1} = \frac{(x_n - \frac{\alpha}{A-1}) + (\gamma - \frac{\alpha}{A-1})x_{n-k}}{A + x_{n-k}}, \text{ for } n \geq 0. \quad (9)$$

$$x_{n+1} - \bar{x} = \frac{(x_n - \bar{x}) + (\bar{x} - \gamma)(\bar{x} - x_{n-k})}{A + x_{n-k}}, \text{ for } n \geq 0. \quad (10)$$

$$x_{n+1} - x_n = \frac{(A-1)(\frac{\alpha}{A-1} - x_n) + (\gamma - x_n)x_{n-k}}{A + x_{n-k}}, \text{ for } n \geq 0. \quad (11)$$

If  $A\gamma - \alpha = \gamma$ , then the unique positive equilibrium is  $\bar{x} = \gamma$ , and (7) and (11) change into

$$x_{n+1} - \gamma = -\frac{\gamma - x_n}{A + x_{n-k}}, \text{ for } n \geq 0. \quad (12)$$

$$x_{n+1} - x_n = \frac{(\gamma - x_n)(A - 1 + x_{n-k})}{A + x_{n-k}}, \text{ for } n \geq 0. \quad (13)$$

The following lemma is straightforward consequences of identities (6)-(13).

**Lemma 4.1** *Assume that  $A\gamma - \alpha < \gamma$  and let  $\{x_n\}_{n=-k}^{\infty}$  be a solution of Eq.(1). Then the following statements are true.*

- (i) *If for some  $N \geq 0$ ,  $x_N \leq \frac{\alpha}{A-1}$ , then  $x_{N+1} \leq \frac{\alpha}{A-1}$ ;*
- (ii) *If for some  $N \geq 0$ ,  $x_N \geq \gamma$ , then  $x_{N+1} \geq \gamma$ ;*
- (iii) *If for some  $N \geq 0$ ,  $x_{N-k} > \bar{x}$  and  $x_N < \bar{x}$ , then  $x_{N+1} < \bar{x}$ ;*
- (iv) *If for some  $N \geq 0$ ,  $x_{N-k} < \bar{x}$  and  $x_N > \bar{x}$ , then  $x_{N+1} > \bar{x}$ ;*
- (v) *If for some  $N \geq 0$ ,  $x_N > \frac{\alpha}{A-1}$ , then  $x_{N+1} < x_N$ ;*
- (vi) *If for some  $N \geq 0$ ,  $x_N < \gamma$ , then  $x_{N+1} > x_N$ ;*
- (vii)  $\gamma \leq \bar{x} \leq \frac{\alpha}{A-1}$ .

**Lemma 4.2** *Assume that  $A\gamma - \alpha \geq \gamma$  and let  $\{x_n\}_{n=-k}^{\infty}$  be a solution of Eq.(1). Then the following statements are true.*

- (i) *If for some  $N \geq 0$ ,  $x_N \geq \frac{\alpha}{A-1}$ , then  $x_{N+1} \geq \frac{\alpha}{A-1}$ ;*
- (ii) *If for some  $N \geq 0$ ,  $x_N \geq A\gamma - \alpha$ , then  $x_{N+1} \geq \gamma$ ;*
- (iii) *If for some  $N \geq 0$ ,  $x_{N-k} > \bar{x}$  and  $x_N > \bar{x}$ , then  $x_{N+1} > \bar{x}$ ;*
- (iv) *If for some  $N \geq 0$ ,  $x_{N-k} < \bar{x}$  and  $x_N < \bar{x}$ , then  $x_{N+1} < \bar{x}$ ;*
- (v) *If for some  $N \geq 0$ ,  $x_N < \frac{\alpha}{A-1}$ , then  $x_{N+1} > x_N$ ;*
- (vi) *If for some  $N \geq 0$ ,  $x_N > \gamma$ , then  $x_{N+1} < x_N$ ;*
- (vii)  $\frac{\alpha}{A-1} \leq \bar{x} \leq \gamma$ .

The following results are consequences of Lemma 4.1-4.2.

**Theorem 4.3** *Let  $\{x_n\}_{n=-k}^{\infty}$  be a non-trivial solution of Eq.(1) and  $\bar{x}$  is the unique positive equilibrium point of Eq.(1). Then the following statements are true:*

- (a) *Assume that  $A\gamma - \alpha < \gamma$ . Then except possibly for the first semicycle, every oscillatory solution of Eq.(1) which lies in the invariant interval  $[\gamma, \frac{\alpha}{A-1}]$  has semicycles of length at least  $k+1$ , or of length at most  $k-1$ .*
- (b) *Assume that  $A\gamma - \alpha \geq \gamma$ . Then except possibly for the first semicycle, every oscillatory solution of Eq.(1) which lies in the invariant interval  $[\frac{\alpha}{A-1}, \gamma]$  has semicycles of length at most  $k$ .*

## 5 Global Stability

In this section, we will investigate global stability of the positive equilibrium point  $\bar{x}$  of Eq.(1).

**Theorem 5.1** *Assume that (5) holds and let  $\{x_n\}_{n=-k}^{\infty}$  be a positive solution of Eq.(1). Then every solution of Eq.(1) eventually enters the invariant interval*

(a)  $[\gamma, \frac{\alpha}{A-1}]$  if  $A\gamma - \alpha < \gamma$ ;

(b)  $[\frac{\alpha}{A-1}, \gamma]$  if  $A\gamma - \alpha \geq \gamma$ .

**Proof.**

(a) In view of Lemma 4.1(i) and (ii), we know that  $x_n \geq \gamma$  and  $x_n \leq \frac{\alpha}{A-1}$  for all  $n \geq 1$  and  $[\gamma, \frac{\alpha}{A-1}]$  is an invariant interval of Eq.(1). If there exist an integer  $N$  such that  $x_N \in [\gamma, \frac{\alpha}{A-1}]$ , then  $x_n \in [\gamma, \frac{\alpha}{A-1}]$  for  $n \geq N$ , from which it follows that the result is true. Now assume for the sake of contradiction that terms of  $\{x_n\}$  never enter the invariant interval  $[\gamma, \frac{\alpha}{A-1}]$ , then there are two cases to be considered:

(i) They all lie in the interval  $[0, \gamma]$ .

(ii) They all lie in the interval  $[\frac{\alpha}{A-1}, \infty)$ .

Case(i). Noticing that  $x_1 \leq \gamma$  and  $A\gamma - \alpha < \gamma$  hold, we get

$$x_2 - x_1 = \frac{\alpha - (A-1)x_1 + x_{1-k}(\gamma - x_1)}{A + x_{1-k}} \geq \frac{\alpha - A\gamma + \gamma}{A + x_{1-k}} > 0,$$

from which it follows by induction that the sequence  $\{x_n\}$  is increasing in the interval  $[0, \gamma]$ . Hence,  $\lim_{n \rightarrow \infty} x_n$  exists and  $\lim_{n \rightarrow \infty} x_n \leq \gamma$ , which is a contradiction because Eq.(1) has no equilibrium point in the interval  $[0, \gamma]$ .

Case(ii). According to  $x_1 \geq \frac{\alpha}{A-1}$  and  $A\gamma - \alpha < \gamma$ , we get

$$x_2 - x_1 = \frac{\alpha - (A-1)x_1 + x_{1-k}(\gamma - x_1)}{A + x_{1-k}} \leq \frac{\alpha - \alpha + x_{1-k}(\gamma - \frac{\alpha}{A-1})}{A + x_{1-k}} < 0,$$

from which it follows by induction that the sequence  $\{x_n\}$  is decreasing in the interval  $[\frac{\alpha}{A-1}, \infty)$ . Hence,  $\lim_{n \rightarrow \infty} x_n$  exists and  $\lim_{n \rightarrow \infty} x_n \geq \frac{\alpha}{A-1}$ , which is a contradiction because Eq.(1) has no equilibrium point in the interval  $[\frac{\alpha}{A-1}, \infty)$ .

(b) The proof is similar to (a), so will be omitted.

The proof is complete.  $\square$

**Theorem 5.2** *Assume that (5) holds. Then the positive equilibrium  $\bar{x}$  is a global attractor of Eq.(1).*

**Proof.** The proof is finished by considering the following two cases.

Case(i) When  $A\gamma - \alpha < \gamma$ . By Lemma 3.2(a) and Theorem 5.1(a), we know that Eq.(1) possesses an invariant interval  $[\gamma, \frac{\alpha}{A-1}]$  and every solution of Eq.(1) eventually enters the interval  $[\gamma, \frac{\alpha}{A-1}]$ . Further, it is easy to see that  $f(u, v)$  increases in  $u$  and decreases in  $v$  in  $[\gamma, \frac{\alpha}{A-1}]$ .

Finally observe that when (5) holds, let  $m, M \in [\gamma, \frac{\alpha}{A-1}]$  is a solution of the system

$$\frac{\alpha + m + \gamma M}{A + M} = m \quad \text{and} \quad \frac{\alpha + M + \gamma m}{A + m} = M,$$

then  $(M - m)(A - 1 + \gamma) = 0$ , which implies that is  $m = M$ . Further, Lemma 4.1 implies that Eq.(1) has a unique equilibrium  $\bar{x} \in [\gamma, \frac{\alpha}{A-1}]$ . Thus, in view of Lemma 1.3, every solution of Eq.(1) converges to  $\bar{x}$ . So the unique positive equilibrium  $\bar{x}$  is a global attractor of Eq.(1).

Case(ii) When  $A\gamma - \alpha \geq \gamma$ . By Lemma 3.2(b) and Theorem 5.1(b), we know that Eq.(1) possesses an invariant interval  $[\frac{\alpha}{A-1}, \gamma]$  and every solution of Eq.(1) eventually enters the interval  $[\frac{\alpha}{A-1}, \gamma]$ . Furthermore, it is easy to see that the function  $f(u, v)$  increases in each of its arguments in the interval  $[\frac{\alpha}{A-1}, \gamma]$  and the system

$$\frac{\alpha + m + \gamma m}{A + m} = m$$

has a unique positive solution in the interval  $[\frac{\alpha}{A-1}, \gamma]$ . Employing Lemma 1.4, we see that Eq.(1) has a unique equilibrium  $\bar{x} \in [\frac{\alpha}{A-1}, \gamma]$  and every solution of Eq.(1) converges to  $\bar{x}$ . Thus the unique positive equilibrium  $\bar{x}$  is a global attractor of Eq.(1).

The proof is complete. □

In view of Theorem 2.1 and 5.2, we have the following result, which solves Conjecture1.1 when condition (5) holds.

**Theorem 5.3** *Assumed that (5) holds. Then the positive equilibrium of Eq.(1) is globally asymptotically stable.*

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