Mathematica Aeterna, Vol. 3, 2013, no. 9, 749 - 751

A simple proof of the important theorem in amenability

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Abstract

In this paper we present a new and simple proof of the important theorem in amenability by the use of paradoxical concept. Principal theorem is that every closed subgroup of the amenable locally compact group G is amenable [Theorem 3.1].

Mathematics Subject Classification: 2010, MSC. 43A07.

Keywords: Paradoxical groups, Amenable locally compact groups, Closed subgroups

1 Introduction

In recent years many mathematicians have studied the amenability of locally compact groups and it's closed subgroups. We recall that, a locally compact group G is amenable if there is a left invariant mean on $L^{\infty}(G)$ [2]. This definition is not always convenient to work with. For example, if H is a closed subgroup of G, then Haar measure on H need not be the restriction to H of Haar measure on G, so that there is, in general, no relation between $L^{\infty}(H)$ and $L^{\infty}(G)$ [3] more particularly awkward if we want to investigate the hereditary properties of amenability. Indeed the amenability is inherited by closed subgroups is surprisingly hard to obtain. In many analysis books for the proof of this fact, Bruhat function, is used [1]. Valker Runde [1] introduced the concept of paradoxical group G in order to show that the locally compact group G, is amenable if and only if G is not paradoxical. In this paper we give a new proof and that seens to be very intersting for obtaining of this fact by using of paradoxical concept the basic tool is a construction very simple to one in Valker Runde ([1], Theorem 1.2.7).

2 Definitions and preliminaries

We begin by making precise what we mean when we say that some set admits a paradoxical decomposition.

Definition 2.1. *Let G be a group which acts on a non-empty set X. Then $E \subset X$ is called G-paradoxical if there are pairwise disjoint subsets $A_1, A_2, \ldots, A_n$, $B_1, B_2, \ldots, B_m$ of E along with $g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_m \in G$ such that $E = \bigcup_{j=1}^{n} g_j \cdot A_j$ and $E = \bigcup_{j=1}^{m} h_j \cdot B_j$.*

Remark 2.2. *We shall simply speak of paradoxical sets (instead of using the slightly leng their adjective G-paradoxical) in the following cases:*
- G acts on itself via multiplication from the left and
- X is a metric space, and G is the group of invertible isometries on X [4].

Proposition 2.3. *Let G be a group which acts on a set X, let H be a subgroup of G, and let $E \subset X$ be H-paradoxical. Then E is G-paradoxical.*

Proof. The proof is straight forward. □

Example 2.4. *The free group $\mathbb{F}_2$ on two generators a, b is paradoxical. For $x \in \{a, b, a^{-1}, b^{-1}\}$, let $W(x) := \{W \in \mathbb{F}_2 : W \text{ starts with } x\}$. Then $\mathbb{F}_2 = \{e_{\mathbb{F}_2}\} \cup W(a) \cup W(b) \cup W(a^{-1}) \cup W(b^{-1})$, where the union is disjoint. Now observe that, for any $W \in \mathbb{F}_2 \backslash W(a)$, we have $a^{-1}W \in W(a^{-1})$, so that $W = a(a^{-1}W) \in aW(a^{-1})$. It follows that $\mathbb{F}_2 = W(a) \cup aW(a^{-1})$ and, similarly $\mathbb{F}_2 = W(b) \cup bW(b^{-1})$, as claimed.*

Proposition 2.5. *For a group G the following are equivalent.*
i) G is not paradoxical;
ii) G is amenable.

Proof. This proposition is an immediate consequence of Tarski's Theorem in [1]. □

According to the Proposition 2.5, it is clear that $\mathbb{F}_2$ is not amenable.

3 The hereditary property of amenability for closed subgroups

The next theorem is a key result in the theory of amenability and we present a new and simple proof for it.

Theorem 3.1. *Let G be an amenable locally compact group and H be a closed subgroup of G, then H is amenable.*

Proof. Suppose that H is not amenable. Then by Proposition 2.5, H is paradoxical and by Proposition 2.3, it is G-paradoxical and therefore G is not amenable. This contradiction proves our assertion. □

The next lemma is an immediate consequence of Example 2.4 and Proposition 2.5 and Theorem 3.1.

Lemma 3.2. *Let G be a locally compact group that contains a closed subgroup isomorphic to $\mathbb{F}_2$, then G is not amenable.*

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Received: October, 2013