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Introduction to d–spaces theory

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Abstract

The aim of this paper is to present the basics of differential spaces theory. In particular differential spaces in a sense of Sikorski are exposed. They are some generalisation of a smooth manifold concept. Except a concise and general exposition to the topic at the introductory level, also some new ideas of gluing two spaces are sketched.

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1 Introduction

The aim of this paper is to present the basics of the differential spaces theory. (Also shortened phrase "d–spaces" is used.) A differential space concept emerged as a generalisation of the manifold concept in 1960s. This kind of generalisation, which will be presented in this paper comes from R. Sikorski [14]. However generalisations in the similar fashion were studied also by other authors (e.g. K. Chen [1], A. Kriegl and P. Michor [8], A. Mallios and E. Rosinger [9], M. Mostow [10], J. Nestruev [11], J. Souriau [16], K. Spallek [17]).

2 Fundamentals

Let M be a set, $M \neq \emptyset$. Let C be a family (finite or infinite) of some real functions on M, i.e. $C := \{f_1, \ldots, f_i, \ldots \mid \forall_i \ f_i : M \to \mathbb{R}\}$.

Definition 2.1. *The weakest topology in which all functions from C are continuous will be called* topology induced by C on M. *This topology will be denoted by τ_C.*

It can simply be proved [14] that:

Fact 2.2. *The base of τ_C is generated by $\emptyset$ and sets $\bigcap_{i=1}^{n}\{p \in M \mid a^i < f^i(p) < b^i\}$, where $n \in \mathbb{N}$, $a^1, \ldots, a^n, b^1, \ldots, b^n \in \mathbb{R}$ and $f^1, \ldots, f^n \in C$.*

The notion of *the continuity of a function* is often misunderstood. Therefore it is briefly summarised in Sec. 3.

Let us consider some function f defined on some subset $A \subset M$.

Definition 2.3. *If $\forall_{p\in A}\ \exists_{B\ni p}\ \exists_{g\in C}\ f|_B = g|_B$ and B is open in topological space (A, τ_A) then f would be called* local C–function. *The set of all local C–functions on a given set $A \subset M$ will be denoted by C_A.*

M. Kreck [7] calls this property *local detectability*. This name nicely resembles some physical consequences. The family of functions may be interpreted as some collection of laboratory machines, used for making measurements (classical, not quantum). Value $f(x)$ of a function f is the result of a measurement done by machine f on the system in a state x [11]. So the family of functions may be understood as the apparatus to recover information about M. It is commonly interpreted that large–scale (global) physics is constructed from local physics (local results of measurements). In other words any statements about global structure are formulated relying on the information taken out form local experiments. The introduced definition proposes rather different way of reasoning. It is the local physics, which has to be consistent with the global one! Local measurements only decode the global information. More discussion about this philosophical consequences can be found e.g. in [5].

Example 2.4. *Notice that for function $f(x) = \frac{1}{x}$ and set $M = (0, 1) \in \mathbb{R}$ it happens that $f \in C^\infty(M)$, but $f \notin C^\infty(\mathbb{R})|_M$. However $f(x) \in (C^\infty(\mathbb{R})|_M)_M$.*

Fact 2.5. *For a set M and families of real functions on this set, C and D, it happens that:*

- $C \subset C_M$,
- $C \subset D \Rightarrow C_M \subset D_M$,
- $(C_M)_M = C_M$.

Sometimes the above properties are summarised in stating that mapping every C to C_M is an algebraic closure in set of all families of real functions on M [18].

Proof. These properties can be checked very easily from the definition. □

Definition 2.6. Superposition closure *of a family of functions C, denoted* sc$C$*, is defined as*

$$\mathrm{sc}C := \{\omega \circ (f_1, \ldots, f_n) \mid n \in \mathbb{N} \ , \ \omega \in C^\infty(\mathbb{R}^n) \ , \ f_1, \ldots, f_n \in C\} \quad .$$

Fact 2.7. *For a set M and families of real functions on this set, C and D, it happens that:*

- $C \subset \mathrm{sc}C \quad ,$
- $C \subset D \Rightarrow \mathrm{sc}C \subset \mathrm{sc}D \quad ,$
- $\mathrm{sc}(\mathrm{sc}C) = \mathrm{sc}C \quad .$

Sometimes the above properties are summarised in stating that mapping every C to sc$C$ *is an algebraic closure in set of all families of real functions on M [18].*

Proof. These properties can be checked very easily from the definition. □

Definition 2.8. *C is called* differential structure *on M if it is closed with respect to localisation ($C = C_M$) and closed with respect to superposition with smooth Euclidean functions ($C =$* sc$C$*).*

Definition 2.9. *A pair (M, C) such that M is an arbitrary set, and C is a family of functions such that $C = (\mathrm{sc}C)_M$ is called a* differential space*.*

Definition 2.10. *If $C_0 := \{f_1, \ldots, f_n\}$ is some family of real functions on M and $C = (\mathrm{sc}C_0)_M$ then the pair (M, C) would be called* differential space generated by C_0*. It is denoted by $C = genC_0$. Functions $f_1, \ldots, f_n$ are called* generators *then.*

Definition 2.11. *If C_0 consists of finite number of functions, then (M, C) is called* finitely generated*.*

Now let us consider two differential spaces (M, C) and (N, D).

Definition 2.12. *Mapping $F : M \to N$ would be called* smooth *if*

$$\forall_{f \in D} \ f \circ F \in C \quad .$$

It can be simply proved that in order to verify smoothness one does not have to check all functions from $D = genD_0$. It is enough to check smoothness on generators from D_0.

Definition 2.13. *F would be called* diffeomorphism *if it is bijective and both F and F^{-1} are smooth.*

Having in mind the remark about physical systems, the diffeomorphic differential spaces gives rise to physically the same systems. I.e. consisting of equivalent states and possible outcomes of measurements on them.

In case there exist some fixed $n \in \mathbb{N}$ and a countable (or finite) covering $\{A\}_{i\in I}$ of M such that for all $i \in I$ there exists diffeomorphism $F_i : (A_i, C_{A_i}) \to (\mathbb{R}^n, C^\infty(\mathbb{R}^n))$ then (M, C) is called a *manifold.* This definition is equivalent to the classical definition of a smooth manifold [14]. However it is interesting that differential calculus and differential geometry can be studied also on differential spaces (as the name suggests), which are not smooth manifolds. This is why differential spaces are generalisation of a classical manifold concept.

It is worth to notice that in Relativity spacetime is modelled by 4–dimensional, smooth, connected manifold equipped with Lorentzian metric. In [4] it is shown that (under some assumptions) differential space may be equipped with Lorentzian metric. However if differential space is not a manifold, then the equivalence principle is not valid. I.e. locally spacetime is not Minkowskian [5].

Moreover let us notice that classically any function $f \in C^\infty(\mathbb{R}^n)$ is called *smooth.* Therefore if one considers some differential space (M, C), which might not be a manifold, the family of functions C can be treated as some analogue of family of classically smooth functions. Therefore it can be introduced

Definition 2.14. *If (M, C) is a differential space, then any $f \in C$ is called a* smooth *function.*

Classically smooth functions are also smooth in the category of differential spaces. Despite this fact a function can be smooth in the category of differential spaces and not be such classically. From now on smoothness will be understood in the category of differential spaces (if not stated otherwise). This kind of smoothness may be also called ”in a sense of Sikorski”.

Example 2.15. *Let $C := gen\{\pi\}$, where π denotes the projection on $\mathbb{R}$. Then $(\mathbb{R}, C)$ is diffeomorphic to $(\mathbb{R}, C^\infty(\mathbb{R}))$. Let $\widetilde{C} := gen\{\pi, |x|\}$. Then $(\mathbb{R}, \widetilde{C})$ and $(\mathbb{R}, C^\infty(\mathbb{R}))$ are not diffeomorphic. Although it is consistent with the definition to say that $|x|$ is smooth (even in 0) when considered on the differential space $(\mathbb{R}, \widetilde{C})$.*

Example 2.16. *Consider the usual topology on $M = \mathbb{R}$. Let C consist of all continuous real functions on $\mathbb{R}$, i.e. $C = C^0(\mathbb{R})$. Then (M, C) is a differential space.*

Definition 2.17. *It is said that (M, C) has a* Hausdorff property *if for each $p, q \in M, p \neq q$ there exists some function $f_{p,q} \in C$ such that $f_{p,q}(p) \neq f_{p,q}(q)$.*

Fact 2.18. *The above definition is equivalent to the classical one.*

Proof. If there is a function $h_{p,q} \in C$ such that $h_{p,q}(p) \neq h_{p,q}(q)$, then take $\epsilon = \frac{1}{2}|h_{p,q}(p) - h_{p,q}(q)|$. Then $(h_{p,q}(p) - \epsilon\ ,\ h_{p,q}(p) + \epsilon) \cap (h_{p,q}(q) - \epsilon\ ,\ h_{p,q}(q) + \epsilon) = \emptyset$. Moreover $p \in A := h_{p,q}^{-1}((h_{p,q}(p) - \epsilon\ ,\ h_{p,q}(p) + \epsilon))$ and $q \in B := h_{p,q}^{-1}((h_{p,q}(q) - \epsilon\ ,\ h_{p,q}(q) + \epsilon))$. Of course $A, B \in \tau_C$ and $A \cap B = \emptyset$, so classical definition of Hausdorff property is satisfied.

On the other hand assume that classical definition of Hausdorff property is satisfied. Let assume that there are some $p, q \in M$ such that there is no $f \in C$ for which $f(p) \neq f(q)$. Then from Fact 2.2 it is clear that for each $A, B \in \tau_C$ if $p \in A$ and $q \in B$ then also $p, q \in A \cap B$. So there are no disjoint neighbourhoods of p and q. Contradiction. □

In order to check Hausdorff property it is not necessary to consider all functions from C. It is enough to check the behaviour of functions from C_0, where $C = genC_0$.

Definition 2.19. *Let (M, C) be a differential space generated by*

$$C_0 = \{f_1, f_2, \ldots, f_n\} \quad .$$

Let $F : M \to \mathbb{R}^n$ be such that $F(p) := (f_1(p), \ldots, f_n(p))$. Then F is called a generator embedding.

The generator embedding is a nice tool, which allows to "see" particular differential space in the familiar Euclidean space. For example it simply and nicely allows to see how modifying generators changes the differential space. This will be presented in further examples. As far as now, it will be proved that:

Theorem 2.20. *If (M, C) is generated by C_0 and has Hausdorff property then generator embedding*

$$F : (M, C) \to (F(M), C^\infty(\mathbb{R}^n)_{F(M)})$$

is a diffeomorphism.

Proof. Indeed F and F^{-1} are smooth. It is enough to check the smoothness on generators. $C^\infty(\mathbb{R}^n)_{F(M)}$ is generated by the projections

$$\pi_1|_{F(M)}, \ldots, \pi_n|_{F(M)} \quad ,$$

therefore it is enough to show that $\pi_i|_{F(M)} \circ F \in C$ for all $i = 1, \ldots, n$. From definition of F we see that $(\pi_i|_{F(M)} \circ F)(p) = \pi_i|_{F(M)}(f_1(p), \ldots, f_n(p)) = f_i(p)$. So globally $\pi_i|_{F(M)} \circ F = f_i \in C$. Due to Hausdorff property of (M, C) for each $q \in F(M)$ exists precisely one $p \in M$ such that $F(p) = q$. Then $F^{-1}(q) = p$ is an inverse mapping. But $f_i(F^{-1}(q)) = \pi_i|_{F(M)}(q)$. So $f_i \circ F^{-1} = \pi_i|_{F(M)} \in C^\infty(\mathbb{R}^n)_{F(M)}$. So F^{-1} is smooth. Finally we obtain that F is injective because of Hausdorff property. □

The symbol $C^\infty(\mathbb{R}^n)_{F(M)}$ may be a bit misleading when compared to the notation introduced in Def. 2.3. However it is used due to historical reasons. To avoid misunderstandings:

Definition 2.21.

$$C^\infty(\mathbb{R}^n)_{F(M)} := (C^\infty(\mathbb{R}^n)|_{F(M)})_{F(M)} = (\mathrm{sc}\{\pi_1|_{F(M)}, \ldots, \pi_n|_{F(M)}\})_{F(M)} \quad .$$

Moreover if (M, C) has not Hausdorff property, we can introduce some equivalence relation in order to impose the required property. That will be described in Sec. 4.

Definition 2.22. *A* tangent vector to (M, C) at a point $p \in M$ *is any linear mapping*

$$v_p \ : \ C \to \mathbb{R} \quad ,$$

such that the Leibniz rule is satisfied, i.e.:

$$\forall_{f_1, f_2 \in C} \ v_p(f_1 \cdot f_2) = v_p(f_1) \cdot f_2(p) + f_1(p) \cdot v_p(f_2) \quad .$$

Of course C is a $\mathbb{R}$–algebra (with pointwise operations). (Moreover (M, C) is also a *ringed space* in a language of sheaves.) Therefore a tangent vector is a derivation of the algebra C in a point p.

To avoid any misunderstandings a definition of a $\mathbb{K}$–*algebra* is given:

Definition 2.23. *Let $\mathbb{K}$ be a field, A be a vector space over $\mathbb{K}$ and*

$$\cdot \ : \ A \ \cdot \ A \to A$$

a binary operation. Assume that $\forall \ a, b, c \in A \ , \ k, l \in \mathbb{K}$ the following identities hold:

- $(a + b) \cdot c = a \cdot c + b \cdot c$,
- $a \cdot (b + c) = a \cdot b + a \cdot c$,
- $(ka) \cdot (lb) = (kl)(a \cdot b)$.

Then A is called an algebra over $\mathbb{K}$ *or* a $\mathbb{K}$–algebra.

Definition 2.24. *The linear space of all tangent vectors to (M, C) at $p \in M$ is called a* tangent space *and is denoted by T_pM.*

Definition 2.25. *We define $df \ : \ TM \to \mathbb{R}$ by requiring that*

$$\forall_{v_p \in TM, f \in C} \ (df)(v_p) = v_p(f) \quad .$$

The differential structure generated by $\{f \circ \pi \mid f \in C\} \cup \{df \mid f \in C\}$ on TM is denoted by TC.

Definition 2.26. *A mapping* $V : M \to TM$*,* $V : p \mapsto V_p$ *is called a* tangent vector field to (M, C)*. (Sometimes the word "tangent" will be omitted.) It is called* smooth*, if for all* $\alpha \in C$*,* f *defined as* $f : p \mapsto V_p(\alpha)$ *belongs to* C*.*

All smooth vector fields tangent to (M, C) constitute a C–module denoted by $\mathfrak{X}(M)$. Every vector field $X \in \mathfrak{X}(M)$ is a smooth section of $\pi : TM \to M$.

Definition 2.27. d *is called a* derivation of a $\mathbb{K}$–algebra A*, if* $d : A \to A$ *and for all* $a, b \in A$*,* $k \in \mathbb{K}$ $d(a + b) = d(a) + d(b)$*,* $d(ka) = kd(a)$ *and* $d(ab) = ad(b) + d(a)b$*.* $\mathrm{Der}(A)$ *denotes all derivations of an algebra* A*.*

Theorem 2.28. *Smooth vector fields to* (M, C) *are in* $1 - 1$ *correspondence with derivations of a* $\mathbb{R}$*–algebra* C*.*

Proof. Consider a mapping $\partial_V : C \to C$ given by the formula $\partial_V(\alpha) = V(\alpha)$, where $\alpha \in C$ and V is a smooth vector field. Let $\beta \in C$ and $p \in M$. (V is fixed, the other objects are variable.)

$$\begin{aligned} \partial_V(\alpha\beta)(p) &= V(\alpha\beta)(p) = V_p(\alpha\beta) = V_p(\alpha)\beta(p) + \alpha(p)V_p(\beta) \\ &= (V(\alpha)\beta + \alpha V(\beta))(p) \end{aligned}$$

Therefore

$$\partial_V(\alpha\beta) = V(\alpha)\beta + \alpha V(\beta) = \partial_V(\alpha)\beta + \alpha\partial_V(\beta) \quad .$$

Similarly it is easy to check that $\partial_V(\alpha + \beta) = \partial_V(\alpha) + \partial_V(\beta)$ and $\partial_V(k\alpha) = k\partial_V(\alpha)$ for an arbitrary $k \in \mathbb{R}$.

So every smooth vector field constitutes a derivation of a $\mathbb{R}$–algebra C.

Consider now a derivation $\partial : C \to C$ and the mapping V^∂ defined by the formula $V^\partial_p(\alpha) = (\partial\alpha)(p)$, where $\alpha \in C$, $p \in M$. (∂ is fixed and the other objects are variable.) For an arbitrary $p \in M$

$$\begin{aligned} V^\partial_p(\alpha\beta) &= (\partial(\alpha\beta))(p) = (\partial(\alpha)\beta + \alpha\partial(\beta))(p) = (\partial\alpha)(p)\beta(p) + \alpha(p)(\partial\beta)(p) \\ &= V^\partial_p(\alpha)\beta(p) + \alpha(p)V^\partial_p(\beta) \quad . \end{aligned}$$

Similarly it is easy to check that $V^\partial_p(\alpha + \beta) = V^\partial_p(\alpha) + V^\partial_p(\beta)$ and $V^\partial_p(k\alpha) = kV^\partial_p(\alpha)$ for an arbitrary $k \in \mathbb{R}$. So $V^\partial : M \to TM$, because it has just been proved that V^∂_p is a tangent vector to (M, C) at point p. Of course $V^\partial(\alpha) = \partial\alpha$. $V^\partial(\alpha) \in C$, because $\partial\alpha \in C$, so V^∂ is a smooth vector field.

Finally the above means that every derivation of $\mathbb{R}$–algebra C constitutes a smooth tangent vector field to (M, C).

Resuming both parts of this proof, it can be concluded that

$$\mathfrak{X}(M) \cong \mathrm{Der}(C) \quad .$$

□

Definition 2.29. *A k–*form on (M, C) *is any k–linear mapping*

$$\theta \ : \ TM \times \cdots \times TM \to \mathbb{R} \quad ,$$

such that

$$\theta_p := \theta_{|T_pM \times \cdots \times T_pM} \to \mathbb{R}$$

is k–linear for every $p \in M$.

Definition 2.30. *A k–form is a* differential k–form*, if it is skew–symmetric.*

Definition 2.31. *A* metric *is a symmetric, nondegenerate 2–form on (M, C).*

3 Continuity of a function

In many books the notion of continuity of a function is taken to be equivalent to the fact that the graph of the function is a single, unbroken, with no "gaps". The well known "example" is $f(x) = \frac{1}{x}$. This kind of breakage of the law of continuity was formulated by Arbogast (Louis François Antoine Arbogast (1759 – 1803): a French mathematician, criticised Euler's notion of a function.) He was using the term *discontiguos*.

The notion of a function was evolving through the time. For example for Euler the function must have been written by a single formula. The currently agreed formal definition of a function states that it is a triple (D, C, f), where D is *a domain*, C is *a codomain* and f is some assignment rule. I.e. f assigns for each element from D exactly one element from C. Equivalently f is a relation on (a subset of) $D \times C$ such that for all $d \in D$ there exists $c \in C$ satisfying $(d, c) \in f$. Let us notice that due to this definition two functions are equal if and only if all three elements from their definitions coincide.

In our case we consider some topologies on D and C. Then it may be defined that

Definition 3.1. (D, C, f) *is* continuous at $d \in D$*, if for every open set $O \in C$ containing $f(d)$ there exists an open set, containing d, completely mapped by f to O.*

Example 3.2 ([2], [15])**.** *There is no sense in considering continuity of $(\mathbb{R}\backslash\{0\}, \mathbb{R}, \frac{1}{x})$ in 0, because this function is not defined in point 0. Because it is continuous in every point of its domain, it is a continuous function.*

4 Gluing relation

Definition 4.1. *Let (M, C) be generated by C_0 and denote by ρ_{C_0} a relation such that*

$$p \ \rho_{C_0} \ q \Leftrightarrow \forall_{f \in C_0} \ f(p) = f(q) \quad .$$

Then this relation is called a gluing relation.

We glue points which are the same for generators and we obtain quotient differential space $(M/\rho_{C_0}, C/\rho_{C_0})$.

Definition 4.2. *Differential structure* C/ρ_{C_0} *consists of*

$$\{\varphi : M/\rho_{C_0} \to \mathbb{R} \mid \varphi \circ \pi_{\rho_{C_0}} \in C\} \quad ,$$

where $\pi_{\rho_{C_0}} : M \to M/\rho_{C_0}$ *is a projection on the equivalence classes.*

It is easy to see that C/ρ_{C_0} is generated by functions φ_i, $i = 1, \dots, n$, where $\varphi_i([p]) := f_i(p)$, $p \in M$, $[p] \in M/\rho_{C_0}$.

Definition 4.3. *Function* $f \in C$ *is called* ρ–consistent *if*

$$\forall_{x,y \in M} \ (x \ \rho \ y \Rightarrow f(x) = f(y)) \quad .$$

All ρ–consistent functions are denoted by C_ρ. If $F : (M, C) \to (N, D)$, then we may introduce $F^\star : D \to C$ defined by the formula $F^\star(g) := g \circ F$, $g \in D$. If $f \in D$ then $F^\star(f) \in C$, so $f \in (F^\star)^{-1}(C)$. Therefore the differential structure $(F^\star)^{-1}(C)$ is maximal one which preserves F smooth, due to the fact $D \subset (F^\star)^{-1}(C)$.

Definition 4.4. *Differential structure* $(F^\star)^{-1}(C)$ *is called* coinduced from the differential structure C on N by the mapping F.

Lemma 4.5. $\pi_\rho^\star|_{C/\rho} : C/\rho \to C_\rho$ *is an isomorphism.*

Proof. $\pi_\rho : (M, C) \to (M/\rho, C/\rho)$, so $\pi_\rho^\star(C/\rho) \subset C$. Moreover $\pi_\rho^\star(C/\rho) = C_\rho$.

First we will show that $C_\rho \subset \pi_\rho^\star(C/\rho)$. Indeed, let $f \in C_\rho$. Then $\forall_{x,y \in M} \ (x \ \rho \ y \Rightarrow f(x) = f(y))$. Let define $\hat{f} : M/\rho \to \mathbb{R}$ by the formula $\hat{f}([x]) := f(x)$, where $[x] := \pi_\rho(x)$. This definition does not depend on class representative. $\hat{f} \in (\pi_\rho^\star)^{-1}(C)$, because $\pi_\rho^\star(\hat{f}) = f$. Indeed $\pi_\rho^\star(\hat{f})(x) = (\hat{f} \circ \pi_\rho)(x) = \hat{f}(\pi_\rho(x)) = \hat{f}([x]) = f(x)$ for all $x \in M$. Therefore $\hat{f} \in C/\rho$, $f = \pi_\rho^\star(\hat{f}) \in \pi_\rho^\star(C/\rho)$.

It will be shown now that $\pi_\rho^\star(C/\rho) \subset C_\rho$. Let $g \in \pi_\rho^\star(C/\rho)$. Then $g \in \pi_\rho^\star((\pi_\rho^\star)^{-1}(C))$. Of course $g \in C$. It will be shown that g is constant on equivalence classes. $g \in \pi_\rho^\star(C/\rho) \Rightarrow g = \pi_\rho^\star(h)$, $h \in C/\rho$, $g = h \circ \pi_\rho$. If $x \ \rho \ y$, then $\pi_\rho(x) = \pi_\rho(y)$. Then $g(x) = g(y)$, so $g \in C_\rho$.

$\pi_\rho : (M, C) \to (M/\rho, C/\rho)$ is "onto". Therefore $\pi_\rho^\star$ is a monomorphism. Indeed $\pi_\rho^\star(j) = 0$, $j \in C/\rho \Rightarrow j \circ \pi_\rho = 0$. Therefore $j([x]) = 0$ for all $x \in M$. So $j = 0$.

As being both monomorphism and epimorphism, $\pi_\rho^\star|_{C/\rho}$ is an isomorphism. □

Example 4.6. *Consider* $(\mathbb{R}, C^\infty(\mathbb{R}))$ *and the relation* ρ

$$x \rho y \Leftrightarrow x - y = 2k\pi \quad , \quad k \in \mathbb{Z} \quad .$$

$C_\rho = \{\omega(\sin x, \cos x) \mid \omega \in C^\infty(\mathbb{R})\} \subset C^\infty(\mathbb{R})$. C_ρ *is generated by* $\{f_1, f_2\}$, *where* $f_1(x) = \sin x$, $f_2(x) = \cos x$. C/ρ *is generated by* $\{\hat{f}_1, \hat{f}_2\}$. *Let* $F := (\hat{f}_1, \hat{f}_2)$. $F : (M/\rho, C/\rho) \to (\mathbb{R}^2, C^\infty(\mathbb{R}^2))$. $F(\mathbb{R}/\rho) = \{(\hat{f}_1(x), \hat{f}_2(x)) \in \mathbb{R}^2 \mid x \in \mathbb{R}\} = \{(\hat{f}_1(x), \hat{f}_2(x)) \in \mathbb{R}^2 \mid [f_1(x)]^2 + [f_2(x)]^2 = 1\} = \{(p,q) \in \mathbb{R}^2 \mid p^2 + q^2 = 1\}$.

Finally, due to recent considerations, in case (M, C) not being Hausdorff:

Theorem 4.7. $(M/\rho_{C_0}, C/\rho_{C_0})$ *and* $(F(M/\rho_{C_0}), C^\infty(\mathbb{R}^n)_{F(M/\rho_{C_0})})$ *are diffeomorphic.*

It will also be useful to introduce a disjoint union concept.

5 Disjoint union

It is reminded that

Definition 5.1. *If some family of sets* $\{A_i \mid i \in I\}$ *is considered then by* disjoint union *we denote the set* $\bigsqcup_{i \in I} A_i := \bigcup_{i \in I} \{(x, i) \mid x \in A_i\}$.

Example 5.2. $A_0 = \{1, 2\}$, $A_1 = \{1\}$. *Then* $A_0 \sqcup A_1 = \{(1,0), (2,0), (1,1)\}$.

Disjoint unions of differential spaces were thoroughly studied by W. Sasin [13]. In case of differential spaces we define

Definition 5.3. *The* disjoint union of differential spaces *is given by the formula*

$$(M_1, C_1) \sqcup (M_2, C_2) := (M_1 \sqcup M_2, C_1 \sqcup C_2) \quad ,$$

where $C_1 \sqcup C_2 := \{f_1 \sqcup f_2 \mid f_1 \in C_1, f_2 \in C_2\}$ *and* $f_1 \sqcup f_2$ *is understood as:*

$$(f_1 \sqcup f_2)(x) := \begin{cases} f_1 & x \in M_1 \\ f_2 & x \in M_2 \end{cases} \quad .$$

Of course the topology on $(M_1 \sqcup M_2, C_1 \sqcup C_2)$ is given by the collection of sets $U_1 \sqcup U_2$, where $U_1 \in \tau_{C_1}$, $U_2 \in \tau_{C_2}$. This topology is denoted by $\tau_{C_1} \sqcup \tau_{C_2}$. It is the weakest topology for which any $f_1 \sqcup f_2 \in C_1 \sqcup C_2$ is continuous on $M_1 \sqcup M_2$.

Let us consider a simple

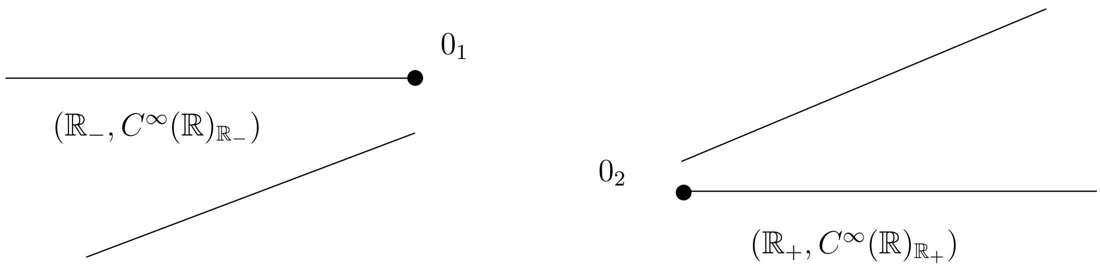

Figure 1: Gluing one–dimensional spaces

Example 5.4. *Let $M := \mathbb{R}_- \sqcup \mathbb{R}_+$ be a disjoint sum. Consider a differential structure C on M. Let C be generated by $\overline{\pi} := \pi|_{\mathbb{R}_-} \sqcup \pi|_{\mathbb{R}_+}$. Of course $\overline{\pi}(0_1) = \overline{\pi}(0_2)$. The quotient space $(M/\rho_{\overline{\pi}}, C/\rho_{\overline{\pi}})$ is diffeomorphic with $(\mathbb{R}, C^\infty(\mathbb{R}))$.*

$$\overline{\pi} : (M/\rho_{\overline{\pi}}, C/\rho_{\overline{\pi}}) \to (\mathbb{R}, C^\infty(\mathbb{R}))$$

In the above example the final differential space occurred to be a classically smooth manifold. It was obtained by taking the disjoint union of two differential spaces: $(\mathbb{R}_-, C_-)$ and $(\mathbb{R}_+, C_+)$, where $C_- = gen\{\pi|_{\mathbb{R}_-}\}$ and $C_+ = gen\{\pi|_{\mathbb{R}_+}\}$. But this result strongly depends on the fact that the generators of the initial spaces were suitably selected. In fact we could have considered different generators, e.g. $(\mathbb{R}_-, C_-)$ and $(\mathbb{R}_+, C_+)$, where $C_- = gen\{-\pi|_{\mathbb{R}_-}\}$ and $C_+ = gen\{\pi|_{\mathbb{R}_+}\}$. Each of these differential spaces is still diffeomorphic with previously considered corresponding differential spaces. Nevertheless $(\mathbb{R}_- \sqcup \mathbb{R}_+, C_- \sqcup C_+) = (\mathbb{R}_- \sqcup \mathbb{R}_+, gen\{|x|\})$ is obtained then. Using generator embedding it can be concluded that this differential space (after implementing gluing relation on it – in order to make it Hausdorff) is diffeomorphic to $(\mathbb{R}_+, C^\infty(\mathbb{R})_{\mathbb{R}_+})$. So it is different from what was obtained in Ex. 5.4.

Definition 5.5. *Such a method of taking suitable generators of initial spaces in order to obtain new glued space will be called* generator gluing technique.

Definition 5.6. *The* disjoint union of tangent spaces to (M, C) *is given by* $TM := \bigsqcup_{p\in M} T_pM$*, with a canonical projection* $\pi : TM \to M$.

It is useful to consider also generalised definitions of some concepts introduced in Sec. 2:

Definition 5.7. *A mapping* $X_1 \sqcup X_2 : X_1 \sqcup X_2 \to T(M_1 \sqcup M_2)$, $X_1 \sqcup X_2 : p \mapsto (X_1 \sqcup X_2)_p$ *is called a* disjoint tangent vector field to $(M_1 \sqcup M_2, C_1 \sqcup C_2)$. *Whereas* $(X_1 \sqcup X_2)_p$ *is understood as below:*

$$(X_1 \sqcup X_2)_p := \begin{cases} X_{1p} & p \in M_1 \\ X_{2p} & p \in M_2 \end{cases} .$$

Then it is easy to prove

Lemma 5.8. *If* $X_1 \in \mathfrak{X}(M_1)$ *and* $X_2 \in \mathfrak{X}(M_2)$ *then* $X_1 \sqcup X_2 \in \mathfrak{X}(M_1 \sqcup M_2)$.

Proof. It must be shown that for arbitrary $p \in M_1 \sqcup M_2$ and arbitrary $f \in C_1 \sqcup C_2$ it is true that $(X_1 \sqcup X_2)_p(f_1 \sqcup f_2) \in C_1 \sqcup C_2$. Indeed, from the definitions:

$$(X_1 \sqcup X_2)_p(f_1 \sqcup f_2) = \begin{cases} X_{1p}(f_1) & p \in M_1 \\ X_{2p}(f_2) & p \in M_2 \end{cases} = \begin{cases} g_1 \in C_1 \\ g_2 \in C_2 \end{cases} = g_1 \sqcup g_2 \in C_1 \sqcup C_2 \quad .$$

□

It is important to remember that smoothness above is not understood classically, but in the differential spaces category.

Definition 5.9. *A* disjoint tangent vector to $(M_1 \sqcup M_2, C_1 \sqcup C_2)$ at a point $p \in M_1 \sqcup M_2$ *is any mapping*

$$(v_1 \sqcup v_2)_p \ : \ C_1 \sqcup C_2 \to \mathbb{R} \quad ,$$

such that

$$(v_1 \sqcup v_2)_p := \begin{cases} v_1 & p \in M_1 \\ v_2 & p \in M_2 \end{cases} \quad ,$$

where v_1 *is some tangent vector to* (M_1, C_1) *and* v_2 *is some tangent vector to* (M_2, C_2).

Of course

Fact 5.10. *A disjoint tangent vector* $(v_1 \sqcup v_2)_p$ *satisfies Leibniz rule.*

Proof. From the definitions

$$(v_1 \sqcup v_2)_p((f_1 \sqcup f_2) \cdot (g_1 \sqcup g_2)) = \begin{cases} v_{1p}(f_1 \cdot g_1) & p \in M_1 \\ v_{2p}(f_2 \cdot g_2) & p \in M_2 \end{cases} \quad .$$

But, as being tangent vectors, both v_{1p} and v_{2p} satisfy Leibniz rule. □

Definition 5.11. *We define* $d(f_1 \sqcup f_2) \ : \ T(M_1 \sqcup M_2) \to \mathbb{R}$ *by requiring that*

$$\forall_{(v_1 \sqcup v_2)_p \in TM_1 \sqcup TM_2, f_1 \sqcup f_2 \in C_1 \sqcup C_2} \ (d(f_1 \sqcup f_2))((v_1 \sqcup v_2)_p) = (v_1 \sqcup v_2)_p(f_1 \sqcup f_2) \quad .$$

The differential structure generated by $\{(f_1 \sqcup f_2) \circ \mathrm{pr}_{\mathrm{M_1 \sqcup M_2}} \mid f_1 \sqcup f_2 \in C_1 \sqcup C_2\} \cup \{d(f_1 \sqcup f_2) \mid f_1 \sqcup f_2 \in C_1 \sqcup C_2\}$ on $T(M_1 \sqcup M_2)$ is denoted by $T(C_1 \sqcup C_2)$. From further considerations (Fact 5.14) it will be clear that this structure is also generated by $(\{f_1 \circ \mathrm{pr}_{\mathrm{M_1}} \mid f_1 \in C_1\} \sqcup \{f_2 \circ \mathrm{pr}_{\mathrm{M_2}} \mid f_2 \in C_2\}) \cup (\{d(f_1) \mid f_1 \in C_1\} \sqcup \{d(f_2) \mid f_2 \in C_2\})$. Moreover $T(C_1 \sqcup C_2) = T(C_1) \sqcup T(C_2)$.

Definition 5.12. *A* disjoint k,l–form on $(M_1 \sqcup M_2, C_1 \sqcup C_2)$ *is any* $k+l$–*linear mapping*

$$\theta \ : \ (TM_1 \times \cdots \times TM_1) \sqcup (TM_2 \times \cdots \times TM_2) \to \mathbb{R} \quad ,$$

such that

$$\theta_p := \begin{cases} \theta_{|T_pM_1 \times \cdots \times T_pM_1} \to \mathbb{R} & p \in M_1 \\ \theta_{|T_pM_2 \times \cdots \times T_pM_2} \to \mathbb{R} & p \in M_2 \end{cases}$$

is

$$\begin{cases} k-\text{linear for every } p \in M_1 \\ l-\text{linear for every } p \in M_2 \end{cases} \quad .$$

Definition 5.13. *A* disjoint metric *is a symmetric, nondegenerate 2,2–form on* $(M_1 \sqcup M_2, C_1 \sqcup C_2)$*, where symmetry and nondegeneracy is understood as for metric in Def. 2.31 for* $\theta_{|T_pM_1 \times T_pM_1} \to \mathbb{R}$ *or* $\theta_{|T_pM_2 \times T_pM_2} \to \mathbb{R}$ *depending on whether* $p \in M_1$ *or* $p \in M_2$*.*

At first sight a little bit of confusion may arise, when considering the above definitions in case of "disjoint generalisation". In particular one may ask whether e.g. Def. 2.26 and Def. 5.7 are consistent if consider $M = M_1 \sqcup M_2$. Happily the below facts in a very natural way confirm the correctness of the introduced definitions.

Fact 5.14. $T(M_1 \sqcup M_2) = TM_1 \sqcup TM_2$

Proof. Indeed, from the definitions:

$$\begin{aligned} T(M_1 \sqcup M_2) &= \bigsqcup_{p \in M_1 \sqcup M_2} T_p(M_1 \sqcup M_2) \\ &= \bigsqcup_{p \in M_1} T_p(M_1 \sqcup M_2) \sqcup \bigsqcup_{p \in M_2} T_p(M_1 \sqcup M_2) \\ &= \bigsqcup_{p \in M_1} T_p(M_1) \sqcup \bigsqcup_{p \in M_2} T_p(M_2) \\ &= TM_1 \sqcup TM_2 \quad . \end{aligned}$$

□

Fact 5.15. *A disjoint union of* k*–form on* (M_1, C_1) *and* l*–form on* (M_2, C_2) *is a disjoint* k,l*–form on* $(M_1 \sqcup M_2, C_1 \sqcup C_2)$ *in a sense of Def. 5.12.*

Proof. Let θ be a k–form on (M_1, C_1) and ω be a l–form on (M_2, C_2). Then

$$(\theta \sqcup \omega)_p = \begin{cases} \theta_{|T_pM_1 \times \cdots \times T_pM_1} \to \mathbb{R} & p \in M_1 \\ \omega_{|T_pM_2 \times \cdots \times T_pM_2} \to \mathbb{R} & p \in M_2 \end{cases} \quad .$$

It is also trivial that

$$\theta \sqcup \omega \ : \ (TM_1 \times \cdots \times TM_1) \sqcup (TM_2 \times \cdots \times TM_2) \to \mathbb{R} \quad .$$

□

Fact 5.16. *Generally*

$$(TM_1 \times TM_1) \sqcup (TM_2 \times TM_2) \neq (TM_1 \sqcup TM_2) \times (TM_1 \sqcup TM_2) \quad .$$

Proof. is trivial. It requires just combinatorial computations, explicitly basing on Def. 5.1 and that of Cartesian product. □

Although Fact 5.16 is very trivial, it is important when studying disjoint k, l–forms. It means that e.g. k, k–form cannot be in general decomposed into k pairwise products of $1, 1$ forms.

Definition 5.17. *A disjoint k, l–form, $\theta \sqcup \omega$, is a* differential disjoint k, l–form, *if θ is skew–symmetric on (M_1, C_1) and ω is skew–symmetric (M_2, C_2).*

The exterior product and exterior derivation may be trivially constructed in case of differential disjoint k, l–forms by considering disjoint union of (classical) differential k–form and l–form. More about differential forms in category of differential spaces may be found in [6].

It is interesting to discuss some of the restrictions that the gluing relation described in Sec. 4 imposes on the disjoint union of differential spaces. For example consider the disjoint union of two differential spaces $(M_1 \sqcup M_2, C_1 \sqcup C_2)$. Assume that this space is not Hausdorff. But it has been showed (Def. 4.1) that $(M_1 \sqcup M_2, C_1 \sqcup C_2)$ may be transformed to possess Hausdorff property. In order to keep clarity assume that there are only two points in $M_1 \sqcup M_2$, for which the gluing relation will hold. Denote them by p_1 and p_2, where $p_1 \in M_1,\ p_2 \in M_2$. The first interesting question is whether a smooth disjoint tangent vector field will be still smooth after the gluing procedure. The answer depends on how the two spaces are glued. The importance of this fact was anticipated in the discussion after Ex. 5.4. At first let us consider that (as in Def. 5.3)

Definition 5.18. $C_1 \sqcup C_2 := \{f_1 \sqcup f_2 \mid f_1 \in C_1, f_2 \in C_2\}$.

I.e. $C_1 \sqcup C_2$ consists of all possible glued pairs of functions from C_1 and C_2. Then:

Theorem 5.19. *A smooth disjoint tangent vector field $X_1 \sqcup X_2 \in \mathfrak{X}(M_1 \sqcup M_2)$ is smooth after the gluing procedure, i.e. $X_1 \sqcup X_2 \in \mathfrak{X}(M_1 \sqcup M_2/\rho_{C_{0,1} \sqcup C_{0,2}})$, if and only if $X_1(p_1) = 0$ and $X_2(p_2) = 0$.*

Proof. First of all notice that $(M_1 \sqcup M_2, C_1 \sqcup C_2)$ is generated by $C_{0,1} \sqcup C_{0,2}$. I.e. $C_{0,1} \sqcup C_{0,2} = \{f_{1i} \sqcup f_{2j} \mid f_{1i} \in C_{0,1},\ f_{2j} \in C_{0,2}\}$, where $C_1 = genC_{0,1}$ and $C_2 = genC_{0,2}$.

In view of Lem. 4.5 instead of $C_1 \sqcup C_2/\rho_{C_{0,1} \sqcup C_{0,2}}$ it may be $(C_1 \sqcup C_2)_{\rho_{C_{0,1} \sqcup C_{0,2}}}$ considered.

If $X_1(p_1) = 0$ and $X_2(p_2) = 0$, then $X_1(p_1) = X_2(p_2)$, so $\forall_{f \in (C_1 \sqcup C_2)_{\rho_{C_{0,1} \sqcup C_{0,2}}}}$

$$X_{1p_1}(f) = X_{2p_2}(f) \quad .$$

But this means that $(X_1 \sqcup X_2)(f) \in (C_1 \sqcup C_2)_{\rho_{C_{0,1} \sqcup C_{0,2}}}$, so $X_1 \sqcup X_2 \in \mathfrak{X}(M_1 \sqcup M_2/\rho_{C_{0,1} \sqcup C_{0,2}})$.

On the other hand assume that $X_1 \sqcup X_2 \in \mathfrak{X}(M_1 \sqcup M_2/\rho_{C_{0,1} \sqcup C_{0,2}})$. It means that $X_{1p_1}(f) = X_{2p_2}(f)$ for an arbitrary $f \in (C_1 \sqcup C_2)_{\rho_{C_{0,1} \sqcup C_{0,2}}}$. In particular let f be such that $f|_{M_1} = f_1$ and $f|_{M_2} = f_2 = f_1(p_1) = const$. Then

$$X_{1p_1}(f_1) = X_{1p_1}(f) = X_{2p_2}(f) = X_{2p_2}(f_2) = X_{2p_2}(const.) = 0 \quad .$$

But from arbitrariness of f_1

$$X_{1p_1} \equiv 0 \quad .$$

Therefore $X_1(p_1) = 0 = X_2(p_2)$. □

It can be seen that the above proof strongly depends on the fact that f such that $f|_{M_1} = f_1$ and $f|_{M_2} = f_2 = f_1(p_1) = const$ belongs to $(C_1 \sqcup C_2)_{\rho_{C_{0,1} \sqcup C_{0,2}}}$. In the above case it was true, because the structure $C_1 \sqcup C_2$ was very large. It consisted of all glued pairs of functions from C_1 and C_2. But it is very interesting to consider some restrictions, i.e. not to glue all available functions from both structures C_1 and C_2, but to properly chose just *generators* of each space C_1 and C_2. Then to glue them and use them to *generate* $(M_1 \sqcup M_2, C)$.

The structure C is now not unique, but depends on which generators were glued. In order to avoid misunderstandings such a structure would be denoted by

$$C_1 \sqcup_{\mathfrak{G}} C_2 \quad .$$

The sign $\mathfrak{G}$ would indicate how the generators from the initial structures and which of them were glued.

It is actually the core of the introduced (see. Def. 5.5) *generator gluing technique* that different pairwise combinations of $f_{1i} \in C_{0,1}$ and $f_{2j} \in C_{0,2}$ may lead to different spaces (see Ex. 5.4).

Definition 5.20. *The* $\mathfrak{G}$–disjoint union of differential spaces *(or shortly: $\mathfrak{G}$–union) is given by the formula*

$$(M_1, C_1) \sqcup_{\mathfrak{G}} (M_2, C_2) := (M_1 \sqcup M_2, C_1 \sqcup_{\mathfrak{G}} C_2) \quad ,$$

where $C_1 \sqcup_{\mathfrak{G}} C_2 := gen\{f_i \mid i = 1, \ldots, k\}$ *and* $f_i = g_{i,1} \sqcup g_{i,2}$ *for some* $g_{i,1} \in C_{0,1}$ *and* $g_{i,2} \in C_{0,2}$. *(*$C_{0,1} = genC_1$ *and* $C_{0,2} = genC_2$.*)*

6 Differential dimension

It is interesting to consider the dimension of a differential space (in the differential spaces category sense).

Definition 6.1. *The* differential dimension *in a point $p \in M$ of a differential space (M, C) is the (classical, i.e. in a sense of a vector space) dimension of a tangent space in this point, i.e.* $\dim T_pM$.

As it is well known the dimension of a (classical) manifold is constant, i.e. the same for each point of the manifold. The above definition does not incorporate any new knowledge in case of a differential space which is a manifold. For example if consider a line as a differential space $(\mathbb{R}, C^\infty(\mathbb{R}))$ or plane $(\mathbb{R}^2, C^\infty(\mathbb{R}^2))$, their differential dimensions are respectively 1 and 2. On the other hand $(\mathbb{R}, \widetilde{C})$ from Ex. 2.15 has differential dimension 1 in each point $p \in (-\infty, 0) \cup (0, \infty)$, but 2 in $p = 0$. So the break up of the property of being a manifold is signalised by the change in differential dimension (in a sense of Def. 6.1). It is also easy to proof that there are no non–zero tangent vectors to the differential space from Ex. 2.16.

Consider the inclusion mapping

$$i_1 : M_1 \to M_1 \sqcup M_2$$

$$i_1(p) = (p, 1)$$

and similarly

$$i_2 : M_2 \to M_1 \sqcup M_2$$

$$i_2(p) = (p, 2) \quad .$$

By i_{1p*} and i_{2q*} we will denote pushforwards of respectively i_1 and i_2 in point $p \in M_1$ and $q \in M_2$.

$$i_{1p*} : T_pM_1 \to T_{i_1(p)}(M_1 \sqcup M_2)$$

$$i_{1p*}(v_p) = (v_{(p,1)}, 1)$$

and similarly

$$i_{2q*} : T_qM_2 \to T_{i_2(q)}(M_1 \sqcup M_2)$$

$$i_{1q*}(v_q) = (v_{(q,2)}, 2) \quad .$$

It can be checked that in a sense of Def. 2.12

Fact 6.2. *The above mappings are smooth.*

Proof. Consider $i_1 : M_1 \to M_1 \sqcup M_2$. It has to be shown that

$$\forall_{f_1 \sqcup f_2 \in C_1 \sqcup C_2} (f_1 \sqcup f_2) \circ i \in C_1 \quad .$$

Indeed, take an arbitrary $p \in M_1$, then

$$((f_1 \sqcup f_2) \circ i_1)(p) = (f_1 \sqcup f_2)(p, 1) = f_1(p) \quad ,$$

so

$$((f_1 \sqcup f_2) \circ i_1)(\ \cdot\ ) = f_1(\ \cdot\ ) \quad .$$

Therefore $(f_1 \sqcup f_2) \circ i_1 \in C_1$, which by Def. 2.12 means that i_1 is smooth.
The remaining three proofs are similar. □

Of course all four above mappings are injective. As a result

$$T(M_1 \sqcup M_2) = \bigoplus_{(p,j),\ j=1,2} i_j T_{(p,j)} M_j \quad .$$

7 Final remarks

The presented theory may be interesting as a branch of pure mathematics, dealing with concepts more general than classical smooth manifolds. But currently there is no agreement which generalisation is the best one. As it has been stated at the beginning, many concepts are explored. Nevertheless the presented concept of generator gluing technique seems to be very useful and more workable than similar competing concepts from other theories. This particular application will be presented with more details in another paper.

Differential spaces in a sense of Sikorski may be also useful when differential geometry needs to be studied in non–smooth cases. Such a situation occurs in many real life engineering problems, and also in theoretical physics (see e.g. [12]). Some examples from different branches of science are sketched e.g. in [3].

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