Mathematica Aeterna, Vol. 3, 2013, no. 9, 771 - 777

Non-traveling wave solutions of Burgers equations with variable coefficients

Zhong Bo Fang*
fangzb7777@hotmail.com
School of Mathematical Sciences,
Ocean University of China,
Qingdao 266100, P.R. China

Lina Zhao
zlngreat@163.com
School of Mathematical Sciences,
Ocean University of China,
Qingdao 266100, P.R. China

Abstract

The present paper deals with families of new exact non-traveling wave solutions of Burgers equations with variable coefficients. By using extended hyperbolic function expansion method, we find two types of non-traveling wave solutions.

Mathematics Subject Classification: 35L25, 35G25.
Keywords: extended hyperbolic function expansion method, Burgers equation, non-traveling wave solution.

1 Introduction

In this paper, we consider the extended Burgers equation:

$$u_t + \alpha(t)\delta(x)uu_x + \beta(t)\gamma(x)u_{xx} = 0, \tag{1}$$

where $\alpha(t)$ and $\beta(t)$ are functions of argument t only, $\delta(x)$ and $\gamma(x)$ are functions of x only. $u = u(x,t) : R \times R \to R$ is an unknown function. The diffusion coefficient $\beta\gamma$ and the convection effect $\alpha\delta$ not only change with time t, but also change with spatial location x.

Burgers equation is very important in the study of solitary wave theory. It appears in many fields, such as fluid mechanics, traffic flows, acoustic transmission and the structure of shock waves, see [1] and the references therein. Many systematic methods are used for studying the nonlinear evolution equations which give rise to new exact solutions, such as Homogeneous balance method [2], the extended $(\frac{G'}{G})$-expansion method [3], Hyperbolic function method [4]. But there is no unified method can be used to deal with all types of nonlinear evolution equations, so it is especially important to select appropriate method to solve exact solutions for specific equations. When $\alpha(t), \beta(t), \delta(x), \gamma(x)$ are all constants, Wang et al. [5] used auxiliary equation method to obtain the hyperbolic tangent type and the tangent type solutions. Zhang [6] used symbolic computation to obtain the two-soliton solution and periodic solution of the Burgers equation. With the help of the Backlund transformation and the Cole-Hopf transformation, Abdul-Majid [7] used singular popular method to solve the (2+1)-dimensional Burgers equation and (2+1)-dimensional higher order Burgers equation, and obtain kink-type solution .

When $\alpha(t), \beta(t), \delta(x), \gamma(x)$ are not all constants, under appropriate conditions through the non-traveling wave transformation $\xi = k(x) + c(t)$, Shi [8] used the hyperbolic function expansion method and homotopy analysis method to obtain the corresponding kink-type solitary wave solutions and periodic solutions. In addition, for exact non-traveling wave solutions, we refer readers to [9,10] and the references therein.

Motivated by the above works, our goal is to construct exact non-travelling wave solutions of Burgers equations (1) with variable coefficients. By using extended hyperbolic function expansion method, we find two types of non-traveling wave solutions.

2 The hyperbolic function expansion method

To determine $u = u(x_1, x_2, x_3, \cdots, t)$ explicitly, we take the following three steps:

Step 1: The general form of nonlinear evolution equations can be written as:

$$F(u, u_t, u_x, u_{xt}, u_{xx}, ...) = 0, \tag{2}$$

where $u = u(x_1, x_2, x_3, ..., t)$ is an unknown function, and F is a polynomial of $u = u(x_1, x_2, x_3, \cdots, t)$.

We use the transformation

$$u(x,t) = u(\xi),\ \xi = \xi(x,t),$$

, we can convert the partial differential equation into standard ordinary differential equations only on the argument ξ.

$$F(u, u', u'', ...) = 0. \tag{3}$$

Step 2: In order to get the exact analytical solutions of the equation, we introduce two basic functions:

$$f = \frac{1}{\cosh(\xi) + r}, \; g = \frac{\sinh(\xi)}{\cosh(\xi) + r}, \tag{4}$$

which are called the expanded functions. They satisfy the following coupling conditions:

$$\frac{df}{d\xi} = -fg, \; \frac{dg}{d\xi} = 1 - g^2 - rf, \; g^2 = 1 - 2rf + (r^2 - 1)f^2. \tag{5}$$

Step 3: Assume that the equation (1) has the following solution:

$$\phi = \sum_{i=0}^{n} a_i f^i + \sum_{i=1}^{n} f^{i-1} g, \tag{6}$$

here $a_0, a_i, b_i (i = 1, 2, ...n)$ are real constants to be determined, n is a positive integer which can be determined by balancing the highest order derivative terms with the highest power nonlinear terms. Then substituting equation (6) into equation (2), and using equation (5), one can simplify the equation which satisfies the following conditions:

I: only having the power of f and g;

II: the power of g is not greater than 1.

Merging the same power of f and g, and taking the coefficients to zero, we can get a set of nonlinear algebraic equations with undetermined coefficients. By solving these equations, we get the exact solutions of nonlinear evolution equations eventually.

Notes that, in this paper, we do not consider the common traveling wave solutions in the form: $u(x,t) = u(\xi), \; \xi = x - \omega t$. In converse, we consider non-traveling wave solutions in the form $\xi = k(t)x + c(t)$ and $\xi = k(x)c(t)$. According to computation, we find two major categories of non-traveling wave solutions and the method we used can be applied to other equations with variable coefficients.

3 Non-traveling wave solutions

We will use the method introduced above to solve the Burgers equation with variable coefficients as follows:

$$u_t + \alpha(t)\delta(x)uu_x + \beta(t)\gamma(x)u_{xx} = 0. \tag{7}$$

According to the homogeneous balance principle for $n = 1$, so the solutions of above equation can be written as:

$$u(\xi) = a_0 + a_1 f(\xi) + b_1 g(\xi). \tag{8}$$

Type I: The transformation is: $\xi = k(t)x + c(t)$.

When $f(\xi)$ and $g(\xi)$ are given by equation (4), substituting equation (8) into equation (7) to simplify the obtained equation with the help of equation (5), and collecting the coefficients of $f^i(\xi)g^j(\xi)(i = 0, 1, 2, 3; j = 0, 1)$, then setting each coefficient to zero, we can derive a set of over-determined ordinary differential equations for a_0, a_1, b_1 and ξ.

$$b_1(a_0 r - a_1 k(t)\delta(x)\alpha(t) + a_1 k^2(t)\beta(t)\gamma(x) + b_1 r(c'(t) + xk'(t)) = 0, \quad (9a)$$

$$b_1(a_0(1-r^2)+3a_1 r)k(t)\delta(x)\alpha(t)-3a_1 rk^2(t)\beta(t)\gamma(x)+b_1(1-r^2)(c'(t)+xk'(t)) = 0, \quad (9b)$$

$$2a_1 b_1(1 - r^2)k(t)\delta(x)\alpha(t) - 2a_1(1 - r^2)k^2(t)\beta(t)\gamma(x) = 0, \quad (9c)$$

$$-a_0 a_1 k(t)\delta(x)\alpha(t) + b_1 rk^2(t)\beta(t)\gamma(x) - a_1(c'(t) + xk'(t)) = 0, \quad (9d)$$

$$-2a_1 k(t)\delta(x)\alpha(t) - 2b_1(1 - r^2)k^2(t)\beta(t)\gamma(x) = 0. \quad (9e)$$

By using Mathematica to simplify the above equations, we can get the following results:

$$a_1 = \pm\sqrt{1 - r^2}b_1, \quad (10a)$$

$$4k(t) = -b_1 \frac{\alpha(t)\delta(x)}{\beta(t)\gamma(x)}, \quad (10b)$$

$$c'(t) = \frac{-a_0 a_1 k(t)\delta(x)\alpha(t) + b_1 rk^2(t)\beta(t)\gamma(x) - a1}{x} k'(t)a_1. \quad (10c)$$

Then by solving $k(t)$ and $c(t)$, the relationship that coefficients in equation (7) is satisfied.

Equation ($10b$) shows that only when $\frac{\delta(x)}{\gamma(x)}$ is a constant it can be satisfied. Assuming the constant is C_1, we get:

$$\delta(x) = C_1\gamma(x),\ k(t) = -b_1 C_1 \frac{\alpha(t)}{\beta(t)}. \quad (11)$$

Substituting equation (6) into equation (5c), it follows that:

$$\frac{c'(t) + xk'(t)}{\gamma(x)} = \frac{-a_0 a_1 k(t)C_1\alpha(t) + b_1 rk^2(t)\beta(t)}{a_1}. \quad (12)$$

The left of equation (12) is a function of x and t, the right is a function of b, then the equation is equivalent to a constant to ensure that the left and right sides of (12) are equivalent, so we have

$$c'(t) + xk'(t) = C_2\gamma(x), \quad (13)$$

$$\frac{-a_0 a_1 k(t)C_1\alpha(t) + b_1 rk^2(t)\beta(t)}{a_1} = C_2. \quad (14)$$

Substituting (11) into (14), we get

$$\frac{(a_0a_1b_1 + b_1^3r)C_1^2\alpha^2(t)}{\beta(t)} = C_2. \tag{15}$$

Assume that $(a_0a_1b_1 + b_1^3r)C_1^2 = C$, then

$$\frac{C\alpha^2(t)}{\beta(t)} = C_2. \tag{16}$$

When $c(t)$ and $k(t)$ are both linear functions, according to equation (13), we know that the above equation can be satisfied, and $\gamma(x)$ is also a linear function.

Assume that

$$c(t) = c_1(t) + c_2,$$
$$k(t) = k_1(t) + k_2,$$

where k_1 and c_1 are both arbitrary non-constant functions, k_2 and c_2 are both arbitrary constants.

Then the non-traveling wave solution of equation

$$u_t + \alpha(t)\delta(x)uu_x + \beta(t)\gamma(x)u_{xx} = 0$$

is:

$$u_1(x,t) = a_0 \pm \frac{\sqrt{1-r^2}b_1}{r + \cosh(k(t)x + c(t))} + \frac{b_1 \sinh(k(t)x + c(t))}{r + \cosh(k(t)x + c(t))}, \tag{17}$$

where $k(t) = k_1(t)+k_2$, $c(t) = c_1(t)+c_2$, k_1, k_2, c_1, c_2 are all arbitrary constants.

Type II: The transformation is $\xi = k(x)c(t)$.

By using Mathematica, we get the following over-determined ordinary differential equations:

$$a_1\delta(x)\alpha(t) + b_1rk(x)c'(t) + c(t)\beta(t)\gamma(x)[a_1c(t)k'(x)^2 + b_1rk''(x) = 0, \tag{18a}$$

$$b_1(1-r^2)k(x)c'(t)+c(t)\beta(t)\gamma(x)[-3a_1rc(t)k'(x)^2+b_1(1-r^2)k''(x)] = 0, \tag{18b}$$

$$b_1\delta(x)\alpha(t) - b_1\beta(t)\gamma(x)k'(x)^2c^2(t) = 0, \tag{18c}$$

$$-a_1k(x)c'(t) + c(t)\beta(t)\gamma(x)[b_1rc(t)k'(x)^2 - a_1k''(x)] = 0. \tag{18d}$$

With the aid of Mathematica, simplifying the above equations, we can get the following results:

$$a_1 = b_1(r^2 - 1), \tag{19a}$$

$$k'(x)^2 = \frac{\delta(x)}{\gamma(x)}, \tag{19b}$$

$$c^2(t) = \frac{\alpha(t)}{\beta(t)}. \tag{19c}$$

According to $k(x)$ and $c(t)$ satisfying:

$$-(r^2-1)k(x)c'(t) + c(t)\beta(t)\gamma(x)(rc(t)k'(x)^2 - (r^2-1)k''(x)) = 0,$$

next, we discuss all the possible values of $k(x)$ and $c(t)$.

Case 1: $k(x)$ is a linear function.

We can get the condition that $c(t)$ satisfies: $c(t) = \int \alpha(t)dt$.

So the solution of equation $u_t + \alpha(t)\delta(x)uu_x + \beta(t)\gamma(x)u_{xx} = 0$ is:

$$u_2(x,t) = a_0 \pm \frac{b_1(r^2-1)}{r+\cosh(k(x)c(t))} + \frac{b_1\sinh(k(x)c(t))}{r+\cosh(k(x)c(t))}, \tag{20}$$

where $k(x) = k_1x + k_2$, $c(t) = \int \alpha(t)dt$, k_1 is a arbitrary non-zero constant, k_2 is a arbitrary constant.

Case 2: $k(x)$ is a trigonometric function.

Assume that $k(x) = k_1 \sin x$, so $c(t)$ is a linear function, and $c(t) = c_1(t)+c_2$. Then the solution of equation $u_t + \alpha(t)\delta(x)uu_x + \beta(t)\gamma(x)u_{xx} = 0$ is:

$$u_3(x,t) = a_0 \pm \frac{b_1(r^2-1)}{r+\cosh(k(x)c(t))} + \frac{b_1\sinh(k(x)c(t))}{r+\cosh(k(x)c(t))}, \tag{21}$$

where $k(x) = k_1\sin(x)$, $c(t) = c_1(t) + c_2$, here k_1, c_1 are arbitrary non-zero constants, c_2 is a arbitrary constant.

Case 3: $k(x)$ is a exponential function ($\gamma(x)$ is not a constant).

Assume that $k(x) = k_1e^x$, so $c(t)$ is a linear function, and $c(t) = c_1(t) + c_2$. Then the solution of equation $u_t + \alpha(t)\delta(x)uu_x + \beta(t)\gamma(x)u_{xx} = 0$ is:

$$u_4(x,t) = a_0 \pm \frac{b_1(r^2-1)}{r+\cosh(k(x)c(t))} + \frac{b_1\sinh(k(x)c(t))}{r+\cosh(k(x)c(t))}, \tag{22}$$

where $k(x) = k_1e^x$, $c(t) = c_1t+c_2$, k_1, c_1 are both arbitrary non-zero constants, c_2 is a arbitrary constant.

4 Conclusion

In this paper, we use the hyperbolic function expansion method to find the exact non-traveling wave solutions of Burgers equation (1) with variable coefficients. We do not consider the common traveling wave solutions, but consider the non-traveling wave solutions. Under appropriate conditions, we obtain two categories of non-traveling wave solutions. The method we use can be applied to the evaluation of the exact solution of a large category of nonlinear evolution equations with variable coefficients.

Acknowledgments

This work is supported by the Natural Science Foundation of Shandong Province of China(ZR2012AM018) and the Fundamental Research Funds for the Central Universities(No.201362032). The authors would like to deeply thank all the reviewers for their insightful and constructive comments.

References

[1] P.L. Sachdev, Nonlinear diffusive waves, Cambridge Univ. Press, Cambridge, 1987.

[2] E. Fan, Two new applications of the homogeneous balance method, J. Phys. Lett. A. 265(2000) 353-357.

[3] I. Aslan and Y. Qzis, Analytic study on two nonlinear evolution equations by using the $(\frac{G'}{G})$-expansion method, J. Comput. Math. Appl. 209(2009) 425-429.

[4] Y. Shang and Y. Huang, Wenjun Yuan, The extended hyperbolic functions and new exact solutions to the Zakharov equations, J. Comput. Math. Appl. 200(2008) 110-122.

[5] D.S. Wang and H.B. Li, Jike, The novel solutions of auxiliary equation and their application to the (2+1)-dimensional Burgers equations, Chaos. Solitons and Fractals. 38(2008) 374-382.

[6] Z. Wang and H.Q. Zhang, Many new kinds exact solutions to (2+1)-dimensional Burgers equation and Klein-Gordon equation used a new method with symbolic computation, Appl. Math. Comput. 18(2007) 693-704.

[7] A. Majid and Wazwaz, A study on the (2+1)-dimensional and the (2+1)-dimensional higher-order Burgers equations, Appl. Math. Lett. (2012).

[8] Y.R. Shi, Application of the Function Expansion Method and the Homotopy Analysis Method for Solving Nonlinear Evolution Eeuations, Doctor thesis, Lanzhou University (2007)(Chinese).

[9] S. Guo and Y. Zhou, Auxiliary equation method for the mkdv equation with variable coefficients, Appl. Math. Comput. 217(2010) 1476-1483.

[10] H.N. Xuan, D. Zhang and C. Wang, Families of non-travelling wave solutions to a generalized variable coefficient two-dimensional Kdv equation using symbolic computation, Chaos. Solitons and Fractals. 23(2005) 171-174.