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On the Wreath product of Group $M_{11} wr M_{12}$ by some other groups

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Abstract

In this paper, we will generate the wreath product $M_{11}wr\ M_{12}$ using only two permutations. We will show the structure of some groups containing the wreath product $M_{11}wr\ M_{12}$. The structure of the group constructed is determined in terms of wreath product$(M_{11}wr\ M_{12}))wrC_k$. Some related cases are also included. Also, we will show that S_{132k+1} and A_{132k+1} can be generated using the wreath product$(M_{11}wr\ M_{12})wrC_k$ and a transposition in S_{132k+1} and an element of order 3 in A_{132k+1} . We will also show thatS_{132k+1} and A_{132k+1} can be generated using the wreath product $M_{11}wr\ M_{12}$ and an element of order k+1 .

1 Introduction

Hammas and Al-Amri [1], have shown that A_{2n+1} of degree $2n + 1$ can be generated using a copy of S_n and an element of order 3 in A_{2n+1} . They also gave the symmetric generating set of Groups $A_{kn+1}$and S_{kn+1} using S_n [5] .

Shafee [2] showed that the groups A_{kn+1} andS_{kn+1} can be generated using the wreath product $A_mwr\ S_a$ and an element of order k+1. Also she showed how to generateS_{kn+1} and A_{kn+1}symmetrically using n elements each of order k+1.

In [3], Shafee and Al-Amri have shown that the groupsA_{110k+1} and S_{110k+1} can be generated using the wreath product $L_2(9)wr\ M_{11}$ and an element of order k+1.

The Mathieu groups M_{11} and M_{12} are two groups of the well known simple groups. In [6] as follows

$$M_{11} =< X, Y\ , Z\ |X^{11} = Y^5 = (XZ)^3 = 1, X^Y = X^4 = Y^z = Y^2 > . \quad (1)$$

$$M_{12} =< X, Y\ , Z\ |X^{11} = Y^2 = Z^2 = (XY)^3 = (XZ)^3 = (YZ)^{10} = 1,\ X^2(YZ)^2X = (YZ)^2 > . \quad (2)$$

M_{11} can be generated using two permutations, the first is of order 11 and 4 as follows :

$$M_{11} =< (1, 2, ..., 11)(1, 2, 3, 7, 6)(3, 4)(6, 8)(4, 8, 5, 9, 10) > . \quad (3)$$

M_{12} can be generated using two permutations, the first is of order 11 and 8 as follows :

$$M_{12} =< (1, 2, ..., 11)(1, 2, 3, 7, 6)(4, 8, 5, 9, 10)(1, 12)(2, 11)(3, 6)(4, 8)(5, 9)(7, 10) > . \quad (4)$$

Here we will generate the wreath product $M_{11}wrM_{12}$ using only two permutations and we will show the structure of some groups containing the wreath product $M_{11}wrM_{12}$. The structure of the groups obtained is determined in terms of wreath product $(M_{11}wrM_{12})wrC_k$.

Some related cases are also included. We will show that S_{132k+1} and A_{132k+1} can be generated using the wreath product$(M_{11}wrM_{12})wrC_k$ and a transposition in $S_{132k+1}$and an element of order in A_{132k+1} . We will also show thatS_{132k+1} and A_{132k+1}can be generated using the wreath product $M_{11}wrM_{12}$ and an element of order .

2 PRELIMINARY RESULTS

DEFINITION 2.1.[4] Let A and B be groups of permutations on non empty sets Ω_1 and Ω_2, respectively, where $\Omega_1 \cap \Omega_2 = \phi$. The wreath product of A

and B is denote by $AwrB$ and defined as $AwrB = A^{\Omega_2} \times_\theta B$, i.e., the direct product of $|\Omega_2|$ copies of A and a mapping θ, where $\theta : B \to Aut(A^{\Omega_2})$ is defined by $\theta_y(x) = x^y$, for all $x \in A^{\Omega_2}$. It follows that

$$|AwrB| = (|A|)^{|\Omega_2|}|B|. \tag{5}$$

THEOREM 2.2 [4] *Let G be the group generated by the n-cycle $(1, 2, \cdots, n)$ and the 2-cycle (n, a). If $1 < a < n$, is an integer with $n = am$, then*

$$G \cong S_m wrC_a. \tag{6}$$

.

THEOREM 2.3 [4] *Let $1 \leq a \neq b < n$ be any integers.Let n be an ood integer and let G the group generated by the n-cycle $(1, 2, \cdots, n)$ and the 3-cycle (n, a, b). If hcf $(n, a, b) = 1$, then $G \cong A_n$* .While if n can be even then

$$G \cong S_n. \tag{7}$$

THEOREM 2.4[4] Let $1 \leq a \leq n$ be any integer. Let $G = \prec (1, 2, ..., n), (n, a) \succ$. If $hcf(n, a) = 1.$, then $G \cong S_n$.

THEOREM 2.5 [4] *Let $1 \leq a \neq b < n$ be any integers.Let n be an even integer and let G the group generated by the n-cycle $(1, 2, \cdots, n)$ and the 3-cycle (n, a, b). Then*

$$G \cong A_n. \tag{8}$$

3 THE RESULTS

THEOREM 3.1 *The wreath product $M_{11}wrM_{12}$ can be generated using* two permutations, the first is of order $132k$ *and the second is of order 4.*

Proof: Let $G = \prec X, Y \succ$,Where:$X = (1, 2, 3, ..., 132)$, $Y = (3, 4, 5, 6)(8, 11, 10, 9)(12, 22)$
(13,26,15,24)(14,23,16,25)(17,27)
(18,31,20,29)(19,28,21,30)
(1,12,16,13)(2,9,24,29)(3,21,30,22)
(4,28,7,20)(5,25,18,11)(10,31,17,23)
which is the product of 12 cycles each of order 4 and two of transpositions

.

Let $\alpha_1 = ((XY)^6[X,Y]^5)^{18}$. Then

α_1 = (11, 22, 33, 44, 55, 66, 77,88,99,110,121,132)

which is a cycle of order 12. Let $\alpha_2 = \alpha_1^{-1}X$.

It is easy to show that

α_2= (1, 2, 3, . . . , 11)(12, 13, 14, . . . , 22) . . . (122,123, 124, . . . , 132),

which is the product of 12 cycles each of order 11.

Let:$\beta_1 = (Y^2)^{(XY)^{18}}$= (9, 20)(12, 23)(31, 53)(34, 56),$\beta_2 = \beta_1\ Y^{-1}$=(1, 9, 12, 20)(2, 6)(4, 5) (7, 8)(13, 17)(15, 16)(18, 19)(23, 31, 45, 53)(24, 28)(26, 27)(29, 30)(34, 42)(35, 39)(37, 38)(40, 41)(46, 50)(48, 49)(51, 52)(56, 64)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71) (73, 74),$\beta_3 = (Y^3\beta_2)^2$ =(1, 45)(12, 23),$\beta_4 = \beta_3^{(\alpha_2^{-1}\alpha_1^3)}$ =(11, 44)(55, 66) and $\beta_5 = \beta_4^{\beta_3^{\alpha_2^{-1}}}$ = (11, 132)(44, 55). Let α_3=$\beta_5^{\beta_3^{(\alpha_2^{-1}\alpha_1)}}$. Hence

$\alpha 3$ =(12, 24)(48, 60).

Let $\alpha_4 = YX^{-1}\alpha_3^{-1}X$. We can conclude that

α_4=(1,9)(2,6)(4,5)(7,8)(12,20)(13,17)(15,16)(18,19)(23,31)(24,28) (26,27)(29,30) (34,42)(35,39)(37,38)(40,41)(45,53)(46,50) (48,49) (51,52)(56,64)(57,61)(59,60)(62,63)(67,75) (68,72)(70,71)(73,74),

which is a product of twenty eight transpositions.

Let $K = \prec \alpha_{2,}\alpha_{4\succ}$. Let $\theta : K \to M_{12}$ be the mapping defined by

θ(12i+j) = j, $\forall 1 \le i \le 10$, $\forall\ 1 \le j \le 12$.

Since $\theta(\alpha_2)$ = (1, 2, . . . , 12) and $\theta(\alpha_4)$ = (1, 9)(2, 6)(4, 5)(7, 8), then K$\approxeq$ $\theta(K) = M_{12}$. Let $H_0 = \prec \alpha_1\ ,\ \alpha_3 \succ$. Then $H_0 \approxeq M_{11}$. Moreover, K conjugates H_0 into H_1 , H_1 into H_2 and so it conjugates H_{11} into H_0 , where

$H_i = \prec (i, 11+i, 22+i,\ 33+i, 44+i, 55+i, 66+i, 77+i, 88+i, 99+i, 110+i, 121+i)(i, 11+i)(22+i, 44+i) \succ\ \ \forall\ 0 \le i \le 11$

" . Hence we get $M_{12}wrM_{11} \subseteq G$. On the other hand, since

X =$\alpha_1\alpha_2$ and Y =$\alpha_4\alpha_3^X$ then $G \subseteq M_{12}wrM_{11}$.

Hence $G = M_{12}wrM_{11}$.$\Diamond$

THEOREM 3.2 The wreath product $(M_{12}wrM_{11})wrC_K$ can be generated using two permutations, the first is of order $132k$ and an involution, for all integers $K \succeq 1$.

Proof :

Let $\sigma = (1, 2, ..., 132k)$,

$\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15, 16k)$
$(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34, 42k, 56k, 64k)(35k.39k)$
$(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)$
$(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k)$,

If k=1, then we get the group$M_{12}wrM_{11}$ which can be considered as the trivial wreath product $M_{12}wrM_{11}$ wr<id>. Assume that $k \succ 1$. Let α $=\Pi_{i=0}^{12}\tau^{\sigma^{ik}}$, we get an element δ $=\alpha^{45}$ =(k, 2k, 3k, ..., 132k). Let G_I $=\prec$ δ^{σ^i} ,τ^{σ^i} $\succ$ be the groups acts on the sets Γ_i ={ i, k+i, 2k+i,..., 131k+i }, for all 1≤ i ≤ k . Since $\cap_{i=1}^{k}\Gamma_i = \phi$, then we get the direct product $G_1 \times G_2$ $\times ... \times G_k$,where, by Theorem 3.1 each $G_i \approxeq M_{12}wrM_{11}$. Let $\beta = \delta^{-1}\sigma$ =(1, 2, ..., k)(k+1, k+2, ..., 2k) ... (76k+1, 76k+2, ..., 132k). Let $H = \prec \beta \succ \approxeq C_k$ .H conjugates G_1 into G_2 ,G_2 into G_3 ,... andG_k into G_1. Hence we get the wreath product $(M_{12}wrM_{11})wrC_K \subseteq G$. On the other hand, since $\delta\beta$ = (1, 2, ..., k, k+1, k+2, ..., 2k, ..., 131k+1, 131k+2, ..., 132k) =σ , then $\sigma \in (M_{12}wrM_{11})wrC_K$. Hence $G = \prec \sigma, \tau \succ \approxeq (M_{12}wrM_{11})wrC_K$.◊

THEOREM 3.3 The wreath product $(M_{12}wrM_{11})wrS_K$ can be generated by using three permutations, the first is of order 132k, the second and the third are involutions , for all K≥ 2.

Proof: Let $\sigma = (1, 2, ..., 132k)$,

$\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15, 16k)$
$(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34, 42k, 56k, 64k)(35k.39k)$
$(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)$
$(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k)$, $\mu = (k, a)(2k, k+a)(3k, 2k+ a)...(143k + 142k + a)$.

since by theorem 3.2 $\prec \sigma, \tau \succ \approxeq (M_{12}wrM_{11})wrC_k$ and (1,2,...,k)(k+1,...,2k)...(131+1,...,132k) $\in (M_{12}wrM_{11})wrC_K$ then $\prec (1, ..., k)(k + 1, ..., 2k)...(131k + 1, ..., 132k, \mu \succ \approxeq S_k$. Hence $G = \prec \sigma, \tau, \mu \succ \approxeq (M_{12}wrM_{11})wrS_{k.}$ ◊

COROLLARY 3.4 The wreath product $(M_{12}wrM_{11}))wrA_k$ can be generated by using three permutations, the first is of order 132k, the second is an involution and the third is of order 3, for all odd integers k ■ 3.

Proof : The proof is similar to the previous one. ◊

THEOREM 3.5 The wreath product $(M_{12}wrM_{11}))wr(S_mwrC_a)$ can be generated by using three permutations, the first is of order 132k, the second and the third are involutions, where $k = am$ be any integer with $1 < a < k$.

Proof : Let $\sigma = (1, 2, ..., 132k)$,

$\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15, 16k)$
$(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34, 42k, 56k, 64k)(35k.39k)$
$(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)$
$(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k)$, $\mu = (k, a)(2k, k+a)(3k, 2k+a)...(132k + 131k + a)$.

since by theorem 3.2 $\prec \sigma, \tau \succ \approxeq (M_{12}wrM_{11}))wrC_k$ and (1,...,k)(k+1,...,2k)...(131+1,...,132k) $\in (M_{12}wrM_{11})wrC_K$ then $\prec (1, ..., k)(k + 1, ..., 2k)...(131k + 1, ..., 132k, \mu \succ \approxeq (S_mwrC_a)$. Hence $G = \prec \sigma, \tau, \mu \succ \approxeq (M_{12}wrM_{11}))wr(S_mwrC_a).\Diamond$

THEOREM 3.6 S_{132K+1} and A_{132K+1}can be generated using the wreath product $(M_{12}wrM_{11})wrC_k$ and a transposition in S_{132K+1} for all integers k >1 and an element of order 11 in A_{132K+1} for all odd integrs k > 1.

Proof : Let $\sigma = (1, 2, ..., 132k)$,

$\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15, 16k)$
$(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34, 42k, 56k, 64k)(35k.39k)$
$(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)$
$(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k)$,

$\mu = (132k + 1, 1)$ and $\mu^\setminus = (1, k, 132k + 1)$ be four Permutations, of order 132k,2,2 and 3 respectively.

Let $H =< \sigma, \tau >$ .By theorem 3.2 $H \approxeq (M_{12}wrM_{11})wrC_K$.

Case 1: Let $G =< \sigma, \tau, \mu^\setminus >$.Let $\alpha = \sigma\mu$, then $\alpha = (1, 2, ..., 132k, 132k+1)$ which is a cycle of order 132k+1. By theorem 2.4 $G =< \sigma, \tau, \mu >=< \alpha, \mu >\approxeq S_{132K+1}$.

Case 2: Let $G =< \sigma, \tau, \mu^\setminus >$ .By theorem 2.5 $< \sigma, \mu^\setminus >\approxeq A_{132K+1}$.Since τ is an even Permutation, then $G \approxeq A_{132K+1}$.

THEOREM 3.7 .S_{132K+1} and A_{132K+1}can be generated using the wreath product $M_{12}wrM_{11}$ and an element of order k+1 inS_{132K+1} and A_{132K+1} for all integers k $\geq$ 1.

Proof : Let $G =< \sigma, \tau, \mu >$, Where

$$\sigma = (1, 2, 3, ..., 132)(132(k - (k - 1)) + 1, ..., 132(k - (k - 1)) + 132)$$
$$...(132(k - 1) + 1, ..., 132(k - 1) + 132),$$

$$\tau = (1,9)(2,6)(4,5(7,8)(12,20,23,31)(13,17)(15,16)(18,19)(24,28)$$
$$(26,27)(29,30)(34,42,56,64,)(35,39)(37,38)(40,41)(45,53)(46,50)$$
$$(48,49)(51,52)(57,61)(59,60)(62,63)(67,75)(68,72)(70,71)(73,74)...$$
$$(132(k-1)+1, 132(k-1)+9)...(132(k-1)+73, 132(k-1)+74),$$

and $\mu = (132, 154, ..., 132k, 132k+1)$,Where $k - i. > 0$, be three permutations of order $132, 4$ and $k+1$ respectively.
Let $H =< \sigma, \tau >$ .Define the mapping θ as follows

$$\theta(12(k-i)+j) = j \ \forall 1 \leq j \leq 12$$

Hence $H =< \sigma, \tau >\cong M_{12}wrM_{11}$.Let $\alpha = \mu\sigma$ it is easy to show that $\alpha = (1, 2, ..., 132k, 132k+1)$, Which is acycle of order 132k+1.
Let $\mu\iota = \mu^{\sigma} = (1, 133, ..., 132(k-1)+1, 132k+1)$ and $\beta = [\mu, \mu^{\backslash}] = (1, 132, 132k+1)$. Since $h.c.f(1, 132, 132k+1) = 1$, then by theorem 2.3 $G =< \sigma, \tau, \mu >\cong S_{132K+1}$ or A_{132K+1} depending on whether k is an odd or an even integer respectively.$\Diamond$.

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