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Numerical technique for solving klein-Gordon equation with purely nonlocal conditions

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Abstract

The aim of this paper are to prove existence, uniqueness, and continuous dependence upon the data of solution to a klein gordon equation with purely nonlocal (integral) conditions. The proofs are based by a priori estimate and inversion Laplace transform technique. Numerical results are provided to show the accuracy of the proposed method.

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1 Introduction

The kleain-Gordon equations appear in quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics, nonlinear optics and applied and physical sciences[2, 18, 22] and are of the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + au(x,t) = f(x,t), \quad 0 < x < 1, \quad 0 < t \le T, \qquad (1)$$

with initial conditions

$$u(x,0) = \varphi(x), \quad \frac{\partial u(x,0)}{\partial t} = \psi(x), \quad 0 < x < 1 \tag{2}$$

and the integral conditions

$$\int_0^1 u(x,t)\,dx = 0, \quad \int_0^1 xu(x,t)\,dx = 0, \quad 0 < t \leq T. \tag{3}$$

where f, φ and ψ are known functions. T and a are known positive constants.

Several techniques including finite difference, collocation, finite element, inverse scattering, decomposition and variational iteration using Adomian's polynomials have been used to handle such equations [2, 18, 22]. We apply the Laplace transforme method (LTM) to solve Klein-Gordon equations. Numerical results show the compte reliability of the proposed technique.

2 Preliminaries

We introduce the appropriate function spaces that will be used in the rest of the note. Let H be a Hilbert space with a norm $\|\cdot\|_H$.

Let $L^2(0,1)$ be the standard function space.

Definition 2.1 *(i) Denote by $L^2(0,T;H)$ the set of all measurable abstract functions $u(.,t)$ from $(0,T)$ into H equiped with the norm*

$$\|u\|_{L^2(0,T;H)} = \left(\int_0^T \|u(.,t)\|_H^2\,dt\right)^{1/2} < \infty. \tag{4}$$

(ii) Let $C(0,T;H)$ be the set of all continuous functions $u(\cdot,t) : (0,T) \to H$ with

$$\|u\|_{C(0,T;H)} = \max_{0 \leq t \leq T} \|u(.,t)\|_H < \infty. \tag{5}$$

We denote by $C_0(0,1)$ the vector space of continuous functions with compact support in $(0,1)$. Since such functions are Lebesgue integrable with respect to dx, we can define on $C_0(0,1)$ the bilinear form given by

$$((u,w)) = \int_0^1 \Im_x^m u \cdot \Im_x^m w dx, \qquad m \geq 1, \tag{6}$$

where

$$\Im_x^m u = \int_0^x \frac{(x-\xi)^{m-1}}{(m-1)!} u(\xi,t)\,d\xi; \qquad for\ m \geq 1. \tag{7}$$

The bilinear form (6) is considered as a scalar product on $C_0(0,1)$ for which $C_0(0,1)$ is not complete.

Definition 2.2 *Denote by* $B_2^m(0,1)$, *the completion of* $C_0(0,1)$ *for the scalar product* (6), *which is denoted* $(.,.)_{B_2^m(0,1)}$, *introduced in [6]. By the norm of function u from* $B_2^m(0,1)$, $m \geq 1$, *we understand the nonnegative number :*

$$\|u\|_{B_2^m(0,1)} = \left(\int_0^1 (\Im_x^m u)^2 \, dx\right)^{1/2} = \|\Im_x^m u\| ; \qquad \text{for } m \geq 1. \tag{8}$$

Lemma 2.3 *For all* $m \in N^*$, *the following inequality holds:*

$$\|u\|_{B_2^m(0,1)}^2 \leq \frac{1}{2} \|u\|_{B_2^{m-1}(0,1)}^2 . \tag{9}$$

Proof 2.4 *See [6].*

Corollary 2.5 *For all* $m \in N^*$, *we have the elementary inequality*

$$\|u\|_{B_2^m(0,1)}^2 \leq \left(\frac{1}{2}\right)^m \|u\|_{L^2(0,1)}^2 . \tag{10}$$

Definition 2.6 *We denote by* $L^2(0,T;B_2^m(0,1))$ *the space of functions which are square integrable in the Bochner sense, with the scalar product*

$$(u,w)_{L^2(0,T;B_2^m(0,1))} = \int_0^T (u(.,t), w(.,t))_{B_2^m(0,1)} \, dt. \tag{11}$$

Since the space $B_2^m(0,1)$ *is a Hilbert space, it can be shown that* $L^2(0,T;B_2^m(0,1))$ *is a Hilbert space as well. The set of all continuous abstract functions in* $[0,T]$ *equipped with the norm*

$$\sup_{0 \leq t \leq T} \|u(.,t)\|_{B_2^m(0,1)}$$

is denoted $C(0,T;B_2^m(0,1))$.

Corollary 2.7 *For every* $u \in L^2(0,1)$, *from which we deduce the continuity of the imbedding* $L^2(0,1) \longrightarrow B_2^m(0,1)$, *for* $m \geq 1$.

Lemma 2.8 (***Gronwall Lemma***) *Let* $f_1(t)$, $f_2(t) \geq 0$ *be two integrable functions on* $[0,T]$, $f_2(t)$ *is nondecreasing. If*

$$f_1(\tau) \leq f_2(\tau) + c \int_0^\tau f_1(t) \, dt, \qquad \forall \tau \in [0,T], \tag{12}$$

where $c \in R^+$, *then*

$$f_1(t) \leq f_2(t) \exp(ct), \qquad \forall t \in [0,T]. \tag{13}$$

Proof 2.9 *The proof is the same as that of Lemma 1.3.19 in [17].*

3 Uniqueness and continuous dependence of the solution

Theorem 3.1 *If* $u(x,t)$ *is a solution of problem*(1)−(3) *and* $f \in C((0,1)\times[0,T])$, *then we have a priori estimates:*

$$\|u(.,\tau)\|^2_{L^2(0,1)} \le c_1\left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}dt + \|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{B_2^1(0,1)}\right) \quad (14)$$

$$\left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(\Omega)} \le c_2\left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}dt + \|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{B_2^1(0,1)}\right) \quad (15)$$

where $c_1 = \exp(T)$, $c_2 = (a+2)\exp(T)$ *and* $0 \le \tau \le T$.

Proof 3.2 *Taking the scalar product in* $B_2^1(0,1)$ *of both sides of equation*(1) *with* $\frac{\partial u}{\partial t}$, *and integrating over* $(0,\tau)$, *we have*

$$\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial t^2}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt - \int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial x^2}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt +$$

$$a\int_0^\tau \left(u(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt = \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt. \quad (16)$$

Integrating by parts of the left-hand sid of (16) *we obtain*

$$\frac{1}{2}\|u(.,\tau)\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(\Omega)} - \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} - \|\psi\|^2_{B_2^1(\Omega)} + \frac{a}{2}\|u(.,\tau)\|^2_{B_2^1(\Omega)} -$$

$$\frac{a}{2}\|\varphi\|^2_{B_2^1(\Omega)} = \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt. \quad (17)$$

By the Chauchy inequality, the right-hand side of (16) *is bounded by*

$$\frac{1}{2}\int_0^\tau \|f(.,t)\|^2_{B_2^1(\Omega)}dt + \frac{1}{2}\int_0^\tau \left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{B_2^1(\Omega)} dt. \quad (18)$$

Substitution of (18) *into* (17), *yields*

$$(a+2)\|u(.,\tau)\|^2_{L^2(\Omega)} + \left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(\Omega)} \le \int_0^\tau \|f(.,t)\|^2_{B_2^1(\Omega)}dt +$$

$$\int_0^\tau \left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{B_2^1(\Omega)} dt + (a+2)\|\varphi\|^2_{L^2(\Omega)} + \|\psi\|^2_{B_2^1(\Omega)}. \quad (19)$$

and by Gronwall Lemma, we have a priori estimates(14)*and* (15).

Corollary 3.3 *If problem* (1) − (3) *has a solution, then this solution is unique and depends continously on* (f,φ,ψ).

4 Existence of the Solution

In this section we shall apply the Laplace transform technique to find solutions of partial differential equations, we have the Laplace transform

$$U(x,s) = \int_0^\infty u(x,t)\exp(-st)\,dt, \tag{20}$$

where s is positive reel parameter. Taking the Laplace transforms on both sides of (1) *, we have*

$$-\frac{d^2U(x,s)}{dx^2} + \left(a+s^2\right)U(x,s) = F(x,s) + s\varphi(x) + \psi(x)\ , \tag{21}$$

where

$$F(x,s) = \int_0^\infty f(x,t)\exp(-st)\,dt.$$

Similarly, we have

$$\int_0^1 U(x,s)\,dx = 0, \tag{22}$$

$$\int_0^1 xU(x,s)\,dx = 0, \tag{23}$$

Thus, considered equation is reduced in boundary value problem governed by second order inhomogeneous ordinary differential equation. We obtain a general solution of (21) *as*

$$\begin{aligned} U(x,s) &= -\frac{1}{\sqrt{a+s^2}}\int_0^x \left[F(\tau,s) + s\varphi(\tau) + \psi(\tau)\ \right]\sinh\left(\sqrt{a+s^2}\,[x-\tau]\right)d\tau \\ &\quad + C_1(s)\exp\left(-\sqrt{a+s^2}x\right) + C_2(s)\exp\left(\sqrt{a+s^2}x\right), \end{aligned} \tag{24}$$

where C_1 *and* C_2 *are arbitrary functions of s. Substitution of* (24) *into* (22) -(23)*, we have*

$$C_1(s)\int_0^1 \exp\left(-\sqrt{a+s^2}x\right)dx + C_2(s)\int_0^1 \exp\left(\sqrt{a+s^2}x\right)dx$$

$$= \frac{1}{\sqrt{a+s^2}}\int_0^1 \left[\left[F(\tau,s) + s\varphi(\tau) + \psi(\tau)\right]\int_\tau^1 \sinh\left(\sqrt{a+s^2}\,[x-\tau]\right)dx\right]d\tau, \tag{25}$$

$$C_1(s)\int_0^1 x\exp\left(-\sqrt{a+s^2}x\right)dx + C_2(s)\int_0^1 x\exp\left(\sqrt{a+s^2}x\right)dx$$

$$= \frac{1}{\sqrt{a+s^2}} \int_0^1 \left[\left[F(\tau,s) + s\varphi(\tau) + \psi(\tau) \right] \int_\tau^1 x \sinh\left(\sqrt{a+s^2} [x-\tau] \right) dx \right] d\tau, \tag{26}$$

where

$$\begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}, \tag{27}$$

and

$$\begin{aligned} a_{11}(s) &= \int_0^1 \exp\left(-\sqrt{a+s^2} x \right) dx, \\ a_{12}(s) &= \int_0^1 \exp\left(\sqrt{a+s^2} x \right) dx, \\ a_{21}(s) &= \int_0^1 x \exp\left(-\sqrt{a+s^2} x \right) dx, \\ a_{22}(s) &= \int_0^1 x \exp\left(\sqrt{a+s^2} x \right) dx, \\ b_1(s) &= \frac{1}{\sqrt{a+s^2}} \int_0^1 \left[\left[F(\tau,s) + s\varphi(\tau) + \psi(\tau) \right] \int_\tau^1 \sinh\left(\sqrt{a+s^2} [x-\tau] \right) dx \right] d\tau \end{aligned}$$

$$b_2(s) = \frac{1}{\sqrt{a+s^2}} \int_0^1 \left[\left[F(\tau,s) + s\varphi(\tau) + \psi(\tau) \right] \int_\tau^1 x \sinh\left(\sqrt{a+s^2} [x-\tau] \right) dx \right] d\tau \tag{28}$$

It is possible to evaluate the integrals in (24) *and* (28) *exactly. In general, one may have to resort to numerical integration in order to compute them, however. For example, the Gauss's formula* (25.4.30) *given in Abramowitz and stegun [1] may be employed to calculate these integrals numerically, we have*

$$\begin{aligned} & \int_0^1 \exp\left(\pm\sqrt{a+s^2} x \right) dx \\ \simeq \; & \frac{1}{2} \sum_{i=1}^{N} \omega_i \exp\left(\pm \frac{\sqrt{a+s^2}}{2} [x_i + 1] \right), \end{aligned}$$

$$\begin{aligned} & \int_0^1 x \exp\left(\pm\sqrt{a+s^2} x \right) dx \\ \simeq \; & \frac{1}{2} \sum_{i=1}^{N} \omega_i \left(\frac{1}{2} [x_i + 1] \right) \exp\left(\pm \frac{\sqrt{a+s^2}}{2} [x_i + 1] \right), \end{aligned}$$

$$\int_0^x \left[F(\tau,s)+s\varphi(\tau)+\psi(\tau)\ \right]\sinh\left(\sqrt{a+s^2}\,[x-\tau]\right)d\tau$$
$$\simeq \frac{x}{2}\sum_{i=1}^{N}\omega_i\left[F\left(\frac{x}{2}[x_i+1];s\right)+s\varphi\left(\frac{x}{2}[x_i+1]\right)+\psi\left(\frac{x}{2}[x_i+1]\right)\right]\times$$
$$\times\sinh\left(\sqrt{a+s^2}\left[x-\frac{x}{2}[x_i+1]\right]\right),$$

$$\int_0^1\left[\left[F(\tau,s)+s\varphi(\tau)+\psi(\tau)\right]\int_\tau^1 \sinh\left(\sqrt{a+s^2}\,[x-\tau]\right)dx\right]d\tau$$
$$\simeq \frac{1}{4}\sum_{i=1}^{N}\omega_i\left[F\left(\frac{1}{2}[x_i+1];s\right)+s\varphi\left(\frac{1}{2}[x_i+1]\right)+\psi\left(\frac{1}{2}[x_i+1]\right)\right]\left(1-\frac{1}{2}[x_i+1]\right)\times$$
$$\times\sum_{i=1}^{N}\omega_j\sinh\left(\sqrt{a+s^2}\left[\frac{1}{2}\left[\left(1-\frac{1}{2}[x_i+1]\right)x_j+\left(1+\frac{1}{2}[x_i+1]\right)\right]-\frac{1}{2}(x_i+1)\right]\right),$$

$$\int_0^1\left[\left[F(\tau,s)+s\varphi(\tau)+\psi(\tau)\right]\int_\tau^1 x\sinh\left(\sqrt{a+s^2}\,[x-\tau]\right)dx\right]d\tau$$
$$\simeq \frac{1}{4}\sum_{i=1}^{2N}\omega_i\left[F\left(\frac{1}{2}[x_i+1];s\right)+s\varphi\left(\frac{1}{2}[x_i+1]\right)+\psi\left(\frac{1}{2}[x_i+1]\right)\right]\left(1-\frac{1}{2}[x_i+1]\right)\times$$
$$\times\left(\frac{1}{2}\left[\left(1-\frac{1}{2}[x_i+1]\right)x_j+\left(1+\frac{1}{2}[x_i+1]\right)\right]\right)\times$$
$$\times\sum_{i=1}^{N}\omega_j\sinh\left(\sqrt{a+s^2}\left[\frac{1}{2}\left[\left(1-\frac{1}{2}[x_i+1]\right)x_j+\left(1+\frac{1}{2}[x_i+1]\right)\right]-\frac{1}{2}(x_i+1)\right]\right),\tag{29}$$

where x_i and w_i are the abscissa and weights, defined as

$$x_i : i^{th}\ zero\ of\ P_n(x), \qquad \omega_i\ = 2/\left(1-x_i^2\right)\left[P_n'(x)\right]^2.$$

Their tabulated values can be found in [1] for different values of N.

4.1 Numerical inversion of Laplace transform

Sometimes, an analytical inversion of a Laplace domain solution is difficult to obtain; therefore a numerical inversion method must be used. A nice comparison of four frequently used numerical Laplace inversion algorithms is given by Hassan Hassanzadeh, Mehran Pooladi-Darvish [16]. In this work we use *the Stehfest's algorithm [20] that is easy to implement. This numerical technique was first introduced by Graver [14] and its algorithm then offered by [20].Stehfest's algorithm approximates the time domain solution as*

$$u(x,t) \approx \frac{\ln 2}{t} \sum_{n=1}^{2m} \beta_n U\left(x; \frac{n \ln 2}{t}\right), \tag{30}$$

where, m is the positive integer,

$$\beta_n = (-1)^{n+m} \sum_{k=\left[\frac{n+1}{2}\right]}^{\min(n,m)} \frac{k^m (2k)!}{(m-k)!k!(k-1)!(n-k)!(2k-n)!}, \tag{31}$$

and $[q]$ denotes the integer part of the real number q.

5 Numerical Examples

In this section, we report some results of numerical computations using Laplace transform method proposed in the previous section. These technique are applied to solve the problem defined by $(1)-(3)$ for particular functions f, φ and ψ, and positive constant a. The method of solution is easily implemented on the computer, used Matlab 7.9.3 program.

Example 5.1 *We take*

$$f(x,t) = 0, 0 < x < 1, 0 < t \leq T\ ,\ a = 1, \varphi(x) = 0, 0 < x < 1, \psi(x) = x, 0 < x < 1, \tag{32}$$

in this case exact solution given by

$$u(x,t) = x \sin t, \quad 0 < x < 1, \quad 0 < t \leq T. \tag{33}$$

The method of solution is easily implemented on the computer, numerical results obtained by $N = 8$ in (29) and $m = 5$ in (30), then we compared the exact solution with numerical solution. For $t = 0.10$, $x \in [0.10, 0.90]$, we calculate u numerically using the proposed method of solution and compare it with the exact solution in Table 1.

x	0.10	0.30	0.50	0.70	0.90
u exact	0,009983341	0,029950025	0,049916708	0,069883391	0,089850075
u numerical	0,009983208	0,029958510	0,049915304	0,069905961	0,089857454
error	−0,000013322	0,000283305	−0,000028126	0,000322966	0,000082157

Table 1

Example 5.2 *We take*

$f(x,t) = 2\sin t, 0 < x < 1, 0 < t \leq T, \ a = 1, \varphi(x) = \sin x, 0 < x < 1, \psi(x) = 1, 0 < x < 1,$

in this case exact solution given by

$$u(x,t) = \sin x + \sin t, 0 < x < 1, 0 < t \leq T. \tag{33}$$

For $t = 0.10$, $x \in [0.10, 0.90]$, *we calculate* u *numerically using the proposed method of solution and compare it with the exact solution in Table 2:*

x	0.10	0.30	0.50	0.70	0.90
u exact	$0,199666683$	$0,395353623$	$0,579258955$	$0,744051103$	$0,883160326$
u numerical	$0,199674896$	$0,395256911$	$0,579266229$	$0,744009648$	$0,883415305$
error	$0,000041133$	$-0,000244621$	$0,000012557$	$-0,000055715$	$0,000288712$

Table 2

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