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About Brezis-Merle Problem with Holderian condition.

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Abstract

We consider the following problem on open set Ω of $\mathbb{R}^2$:

$$\begin{cases} -\Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

Under some conditions , we give a quantization analysis and a compactness result for the previous problem.

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1 Introduction and Main Results

We set $\Delta = \partial_{11} + \partial_{22}$ on open set Ω of $\mathbb{R}^2$ with a smooth boundary.

We consider the following problem on $\Omega \subset \mathbb{R}^2$:

$$(P) \begin{cases} -\Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We assume that,

$$\int_\Omega e^{u_i} dy \leq C, \ \ 0 \leq V_i \leq b < +\infty$$

The previous equation is called, the Prescribed Scalar Curvature equation, in relation with conformal change of metrics. The function V_i is the prescribed curvature. Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of this type were studied by many authors, see [1-10]. We can see in [4], different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on V, for example we suppose $V_i \geq 0$ and $V_i \in L^p(\Omega)$ or $V_i e^{u_i} \in L^p(\Omega)$ with $p \in [1, +\infty]$.

Among other results, we can see in [4], the following important Theorem,

Theorem A*(Brezis-Merle [4]).If* $(u_i)_i$ *and* $(V_i)_i$ *are two sequences of functions relatively to the previous problem* (P) *with,* $0 < a \leq V_i \leq b < +\infty$*, then, for all compact set* K *of* Ω*,*

$$\sup_K u_i \leq c = c(a, b, m, K, \Omega) \text{ if } \inf_\Omega u_i \geq m.$$

A simple consequence of this theorem is that, if we assume $u_i = 0$ on $\partial\Omega$ then, the sequence $(u_i)_i$ is locally uniformly bounded. We can find in [4] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i}.

If, we assume V with more regularity, we can have another type of estimates, $\sup + \inf$. It was proved, by Shafrir, see [10], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C\left(\frac{a}{b}\right) \sup_K u_i + \inf_\Omega u_i \leq c = c(a, b, K, \Omega).$$

We can see in [6], an explicit value of $C\left(\frac{a}{b}\right) = \sqrt{\frac{a}{b}}$. In his proof, Shafrir has used the Stokes formula and an isoperimetric inequality, see [2]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see Brézis-Li-Shafrir [3]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [6]. Also, we can see in [7], an extension of the Brezis-Li-Shafrir to compact Riemann surface without boundary. We can see in [8] explicit form, ($8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses, when the solutions blow-up. Here, the notion of isolated blow-up point is used.

In [4], Brezis and Merle proposed the following Problem:

Problem *(Brezis-Merle [4]).If* $(u_i)_i$ *and* $(V_i)_i$ *are two sequences of functions relatively to the previous problem* (P) *with,*

$$0 \leq V_i \to V \text{ in } C^0(\Omega), \int_\Omega e^{u_i} dy \leq C,$$

Is it possible to prove that:

$$\sup_\Omega u_i \leq c = c(C, V, \Omega) \ ?$$

Here, we assume more regularity on V_i, we suppose that $V_i \geq 0$ is C^s (s-holderian) $1/2 < s \leq 1)$. We give the answer where $bC < 24\pi$.

On other hand, in our work we give a complete caracterisation of the blow-up analysis on the boundary.

Our main results are:

Theorem 1.1 *Assume $\Omega = B_1(0)$, and,*

$$u_i(x_i) = \sup_{B_1(0)} u_i \to +\infty.$$

There is a finite number of sequences $(x_i^k)_i, (\delta_i^k), 0 \leq k \leq m$, such that:

$$(x_i^0)_i \equiv (x_i)_i,\ \ \delta_i^0 = \delta_i = d(x_i, \partial B_1(0)) \to 0,$$

and each δ_i^k is of order $d(x_i^k, \partial B_1(0))$.
and,

$$u_i(x_i^k) = \sup_{B_1(0) - \cup_{j=0}^{k-1} B(x_i^j, \delta_i^j \epsilon)} u_i \to +\infty,$$

$$u_i(x_i^k) + 2 \log \delta_i^k \to +\infty,$$

$$\forall\ \epsilon > 0,\ \ \sup_{B_1(0) - \cup_{j=0}^{m} B(x_i^j, \delta_i^j \epsilon)} u_i \leq C_\epsilon$$

$$\forall\ \epsilon > 0,\ \ \limsup_{i \to +\infty} \int_{B(x_i^k, \delta_i^k \epsilon)} V_i e^{u_i} dy \geq 4\pi > 0.$$

If we assume:

$$V_i \to V \text{ in } C^0(B_1(0)),$$

then,

$$\forall\ \epsilon > 0,\ \ \limsup_{i \to +\infty} \int_{B(x_i^k, \delta_i^k \epsilon)} V_i e^{u_i} dy = 8\pi m_k,\ \ m_k \in \mathbb{N}^*.$$

And, thus, we have the following convergence in the sense of distributions:

$$\int_{B_1(0)} V_i e^{u_i} dy \to \int_{B_1(0)} V e^u dy + \sum_{k=0}^{m} 8\pi m'_k \delta_{x_0^k},\ m'_k \in \mathbb{N}^*,\ \ x_0^k \in \partial B_1(0).$$

Theorem 1.2 *Assume that:*

$$\int_{B_1(0)} V_i e^{u_i} dy \leq 4\pi,$$

Then,

$$u_i(x_i) = \sup_{B_1(0)} u_i \leq c = c(b, C),$$

Theorem 1.3 *Assume that, V_i is uniformly $s-$holderian with $1/2 < s \leq 1$, and,*

$$\int_{B_1(0)} V_i e^{u_i} dy \leq 24\pi - \epsilon, \ \epsilon > 0,$$

then we have:

$$\sup_{\Omega} u_i \leq c = c(b, C, A, s, \Omega).$$

where A is the holderian constant of V_i.

2 Proofs of the theorems:

Without loss of generality, we can assume that $\Omega = B_1(0)$ the unit ball centered on the origin.

Here, G is the Green function of the Laplacian with Dirichlet condition on $B_1(0)$. We have (in complex notation):

$$G(x, y) = \frac{1}{2\pi} \log \frac{|1 - \bar{x}y|}{|x - y|}, \ u_i(x) = \int_{B_1(0)} G(x, y) V_i(y) e^{u_i(y)} dy,$$

we write,

$$u_i(x_i) = \int_{\Omega} G(x_i, y) V_i(y) e^{u_i(y)} dx = \int_{\Omega - B(x_i, \delta_i)} G(x_i, y) V_i e^{u_i(y)} dy + \int_{B(x_i, \delta_i)} G(x_i, y) V_i e^{u_i(y)} dy$$

According to the maximum principle, the harmonic function $G(x_i, .)$ on $\Omega - B(x_i, \delta_i)$ take its maximum on the boundary of $B(x_i, \delta_i)$, we can compute this maximum:

$$G(x_i, y_i) = \frac{1}{2\pi} \log \frac{|1 - \bar{x}_i y_i|}{|x_i - y_i|} = \frac{1}{2\pi} \log \frac{|1 - \bar{x}_i(x_i + \delta_i \theta_i)|}{|\delta_i|} = \frac{1}{2\pi} \log((1 + |x_i|) + \theta_i) < +\infty$$

with $|\theta_i| = 1$.

Thus,

$$u_i(x_i) \leq C + \int_{B(x_i,\delta_i)} G(x_i,y)V_i e^{u_i(y)}dy \leq C + e^{u_i(x_i)} \int_{B(x_i,\delta_i)} G(x_i,y)dy$$

Now, we compute $\int_{B(x_i,\delta_i)} G(x_i,y)dy$
we set in polar coordinates,

$$y = x_i + \delta_i t\theta$$

we find:

$$\int_{B(x_i,\delta_i)} G(x_i,y)dy = \int_{B(x_i,\delta_i)} \frac{1}{2\pi} \log \frac{|1-\bar{x}_i y|}{|x_i - y|} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \delta_i^2 \log \frac{|1-\bar{x}_i(x_i+\delta_i\theta)|}{\delta_i} t dt d\theta =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \delta_i^2 (\log(|1+|x_i|+t\theta|) - \log t) t dt d\theta \leq C\delta_i^2.$$

Thus, we can write, because $u_i(x_i) \to +\infty$,

$$u_i(x_i) \leq C'\delta_i^2 e^{u_i(x_i)},$$

We can conclude that:

$$u_i(x_i) + 2\log\delta_i \to +\infty.$$

Now, consider the following function :

$$v_i(y) = u_i(x_i + \delta_i y) + 2\log\delta_i, \quad y \in B(0,1/2)$$

The function satisfies all conditions of the Brezis-Merle hypothesis, we can conclude that, on each compact set:

$$v_i \to -\infty$$

we can assume, without loss of generality that for $1/2 > \epsilon > 0$, we have:

$$v_i \to -\infty, \quad y \in B(0,2\epsilon) - B(0,\epsilon),$$

Lemma 2.1 *For all* $1/4 > \epsilon > 0$*, we have:*

$$\sup_{B(x_i,(3/2)\delta_i\epsilon) - B(x_i,\delta_i\epsilon)} u_i \leq C_\epsilon.$$

Proof of the lemma

Let t_i' and t_i the points of $B(x_i, 2\delta_i\epsilon) - B(x_i, (1/2)\delta_i\epsilon)$ and $B(x_i, (3/2)\delta_i\epsilon) - B(x_i, \delta_i\epsilon)$ respectively where u_i takes its maximum.

According to the Brezis-Merle work, we have:

$$u_i(t_i') + 2\log\delta_i \to -\infty$$

We write,

$$u_i(t_i) = \int_\Omega G(t_i, y)V_i(y)e^{u_i(y)}dx = \int_{\Omega - B(x_i, 2\delta_i\epsilon)} G(t_i, y)V_ie^{u_i(y)}dy+$$

$$+\int_{B(x_i, 2\delta_i\epsilon) - B(x_i,(1/2)\delta_i\epsilon)} G(t_i, y)V_ie^{u_i(y)}dy + \int_{B(x_i,(1/2)\delta_i\epsilon)} G(t_i, y)V_ie^{u_i(y)}dy$$

But, in the first and the third integrale, the point t_i is far from the singularity x_i and we soon that the Green function is bounded. For the second integrale, after a change of variable, we can see that this integale is bounded by (we take the supremum in the annulus and use Brezis-Merle theorem)

$$\delta_i^2 e^{u_i(t_i')} \times I_j$$

where I_j is a Jensen integrale (of the form $\int_0^1\int_0^{2\pi}(\log(|1 + |x_i| + t\theta) - \log|\theta_i - t\theta|)tdtd\theta$ which is bounded). We conclude the lemma.

From the lemma, we see that far from the singularity the sequence is bounded, thus if we take the supremum on the set $B_1(0) - B(x_i, \delta_i\epsilon)$ we can see that this supremum is bounded and thus the sequence of functions is uniformly bounded or tends to infinity and we use the same arguments as for x_i to conclude that around this point and far from the singularity, the seqence is bounded.

The process will be finished , because, according to Brezis-Merle estimate, in [4], around each supremum constructed and tending to infinity, we have:

$$\forall\ \epsilon > 0,\ \limsup_{i\to+\infty}\int_{B(x_i,\delta_i\epsilon)} V_ie^{u_i}dy \geq 4\pi > 0.$$

Finaly, with this construction, we have a finite number of "exterior "blow-up points and outside the singularities the sequence is bounded uniformly, for example, in the case of one "exterior" blow-up point, we have:

$$u_i(x_i) \to +\infty,\ \forall\ \epsilon > 0,\ \sup_{B_1(0)-B(x_i,\delta_i\epsilon)} u_i \leq C_\epsilon$$

$$x_i \to x_0 \in \partial B_1(0),\ \forall\ \epsilon > 0,\ \limsup_{i\to+\infty}\int_{B(x_i,\delta_i\epsilon)} V_ie^{u_i}dy \geq 4\pi > 0.$$

We have the following lemma:

Lemma 2.2 *Each δ_i^k is of order $d(x_i^k, \partial B_1(0))$. Namely: there is a positive constant $C > 0$ such that for $\epsilon > 0$ small enough:*

$$\delta_i^k \leq d(x_i^k, \partial B_1(0)) \leq (2 + \frac{C}{\epsilon})\delta_i^k.$$

<u>Proof of the lemma</u>

Now, if we suppose that there is another "exterior" blow-up $(t_i)_i$, we have, because $(u_i)_i$ is uniformly bounded in a neighborhood of $\partial B(x_i, \delta_i \epsilon)$, we have :

$$d(t_i, \partial B(x_i, \delta_i \epsilon)) \geq \delta_i \epsilon$$

If we set,

$$\delta_i' = d(t_i, \partial(B_1(0) - B(x_i, \delta_i \epsilon))) = \inf\{d(t_i, \partial B(x_i, \delta_i \epsilon)), d(t_i, \partial(B_1(0)))\}$$

then, δ_i' is of order $d(t_i, \partial B_1(0))$. To see this, we write:

$$d(t_i, \partial B_1(0)) \leq d(t_i, \partial B(x_i, \delta_i \epsilon)) + d(\partial B(x_i, \delta_i \epsilon), x_i) + d(x_i, \partial B_1(0)),$$

Thus,

$$\frac{d(t_i, \partial B_1(0))}{d(t_i, \partial B(x_i, \delta_i \epsilon))} \leq 2 + \frac{1}{\epsilon},$$

Thus,

$$\delta_i' \leq d(t_i, \partial B_1(0)) \leq \delta_i'(2 + \frac{1}{\epsilon}).$$

Now, the general case follow by induction. We use the same argument for three, four,..., n blow-up points.

We have, by induction and, here we use the fact that u_i is uniformly bounded outside a small ball centered at $x_i^j, j = 0, \ldots, k-1$:

$$\delta_i^j \leq d(x_i^j, \partial B_1(0)) \leq C_1 \delta_i^j, \ j = 0, \ldots, k-1,$$

.

$$d(x_i^k, \partial B(x_i^j, \delta_i^j \epsilon/2)) \geq \epsilon \delta_i^j, \epsilon > 0, \ j = 0, \ldots, k-1,$$

and let's consider x_i^k such that:

$$u_i(x_i^k) = \sup_{B_1(0) - \cup_{j=0}^{k-1} B(x_i^j, \delta_i^j \epsilon)} u_i \to +\infty,$$

take,

$$\delta_i^k = \inf\{d(x_i^k, \partial B_1(0)), d(x_i^k, \partial(B_1(0) - \cup_{j=0}^{k-1} B(x_i^j, \delta_i^j \epsilon/2))\},$$

if, we have,

$$\delta_i^k = d(x_i^k, \partial B(x_i^j, \delta_i^j \epsilon/2)),\ j \in \{0, \ldots, k-1\}.$$

Then,

$$\delta_i^k \leq d(x_i^k, \partial B_1(0)) \leq$$

$$\leq d(x_i^k, \partial B(x_i^j, \delta_i^j \epsilon/2)) + d(\partial B(x_i^j, \delta_i^j \epsilon/2), x_i^j) + d(x_i^j, \partial B_1(0))$$

$$\leq (2 + \frac{C_1}{\epsilon})\delta_i^k.$$

To apply lemma 2.1 for m blow-up points, we use an induction:

We do directly the same approch for t_i as x_i by using directly the Green function of the unit ball.

If we look to the blow-up points, we can see, with this work that, after finite steps, the sequence will be bounded outside a finite number of balls , because of Brezis-Merle estimate:

$$\forall\ \epsilon > 0,\ \limsup_{i \to +\infty} \int_{B(x_i^k, \delta_i^k \epsilon)} V_i e^{u_i} dy \geq 4\pi > 0.$$

Here, we can take the functions:

$$u_i^k(y) = u_i(x_i^k + \delta_i^k y),$$

which are bounded on the boundary of $B(0, \epsilon)$.

Finaly, we can say that, there is a finite number of sequences $(x_i^k)_i, (\delta_i^k), 0 \leq k \leq m$, such that the claim of the first theorem is true.

<u>The work of YY.Li-I.Shafrir, in [8]</u>

With the previous method, we have a finite number of "exterior" blow-up points (perhaps the same) and the sequences tend to the boundary. With the aid of proposition 1 of the paper of Li-Shafrir, in [8], we see that around each exterior blow-up, we have a finite number of "interior" blow-ups. Around, each exterior blow-up, we have after rescaling with δ_i^k, the same situation as around a fixed ball with positive radius. If we assume:

$$V_i \to V \text{ in } C^0(B_1(0)),$$

then,

$$\forall\ \epsilon > 0,\ \ \limsup_{i\to+\infty} \int_{B(x_i^k,\delta_i^k\epsilon)} V_i e^{u_i} dy = 8\pi m_k,\ \ m_k \in \mathbb{N}^*.$$

And, thus, we have the following convergence in the sense of distributions:

$$\int_{B_1(0)} V_i e^{u_i} dy \to \int_{B_1(0)} V e^u dy + \sum_{k=0}^{m} 8\pi m_k' \delta_{x_0^k},\ m_k' \in \mathbb{N}^*,\ \ x_0^k \in \partial B_1(0).$$

Consequence 1: Proof of theorem 2

Assume that:

$$\int_{B_1(0)} V_i e^{u_i} dy \leq 4\pi,$$

Then, if the sequence blow-up, there is one and only one blow-up point and we have:

$$u_i(x_i) = \sup_{B_1(0)} u_i \to +\infty,\ \ u_i(x_i) + 2\log\delta_i \to +\infty,$$

$$\forall\ \epsilon > 0,\ \ \sup_{B_1(0)-B(x_i,\delta_i\epsilon)} u_i \leq C_\epsilon$$

We set,

$$r_i = e^{-u_i(x_i)/2},$$

The blow-up function is locally bounded thus,

$$r_i^2 e^{u_i} \leq C \text{ on } B(x_i, 2r_i).$$

We write:

$$u_i(x_i) = \int_{\Omega - B(x_i,\delta_i\epsilon)} G(x_i,y) V_i e^{u_i(y)} dy + \int_{B(x_i,\delta_i\epsilon)} G(x_i,y) V_i e^{u_i(y)} dy$$

we have:

$$\int_{B(x_i,\delta_i\epsilon)} G(x_i,y) V_i e^{u_i(y)} dy = \int_{B(x_i,\delta_i\epsilon)-B(x_i,2r_i)} G(x_i,y) V_i e^{u_i(y)} dy + \int_{B(x_i,2r_i)} G(x_i,y) V_i e^{u_i(y)} dy$$

We use the maximum principle on $B(x_i,\delta_i\epsilon) - B(x_i,2r_i)$ and the explicit formula of G to prove that:

$$G(x_i,y)\leq C+\frac{1}{2\pi}\log\frac{\delta_i}{r_i}=C+\frac{1}{4\pi}(u_i(x_i)+2\log\delta_i).$$

On $B(x_i,2r_i)$ we use the fact that:

$$r_i^2e^{u_i}\leq C$$

and the explicit formula for G to have:

$$\int_{B(x_i,2r_i)}G(x_i,y)V_ie^{u_i(y)}dy\leq C+\frac{1}{2\pi}\log\frac{\delta_i}{r_i}\int_{B(x_i,2r_i)}V_ie^{u_i(y)}dy.$$

We conclude that:

$$u_i(x_i)\leq C+\frac{1}{4\pi}(u_i(x_i)+2\log\delta_i)\int_{B(x_i,\delta_i\epsilon)}V_ie^{u_i(y)}dy.$$

Our hypothesis on the integrale of $V_ie^{u_i}$ imply that:

$$\log\delta_i\geq -C,$$

in other words, we have uniformly,

$$d(x_i,\partial B_1(0))=\delta_i\geq e^{-C}>0.$$

this contredicts the fact that (x_i) tends to the boundary. The sequence (u_i) is bounded in this case.

We can see that the case:

$$\int_{B_1(0)}V_ie^{u_i}dy\leq 4\pi,$$

is optimal, because Brezis-Merle have proved that, there is a counterexample of blow-up sequence with:

$$\int_{B_1(0)}V_ie^{u_i}dy=4\pi A>4\pi.$$

Consequence 2: using a Pohozaev-type identity, proof of theorem 3

By a conformal transformation, we can assume that our domain $\Omega=B^+$ is a half ball centered at the origin, $B^+=\{x,|x|\leq 1,x_1\geq 0\}$. In this case the normal at the boundary is $\nu=(-1,0)$ and $u_i(0,x_2)\equiv 0$. Also, we set x_i the blow-up point and $x_i^2=(0,x_i^2)$ and $x_i^1=(x_i^1,0)$ respectevely the second and the first part of x_i. Let ∂B^+ the part of the boundary for which u_i and its derivatives are uniformly bounded and thus converge to the corresponding function.

The case of one blow-up point:

Theorem 2.3 *If V_i is s-Holderian with $1/2 < s \leq 1$ and,*

$$\int_\Omega V_i e^{u_i} dy \leq 16\pi - \epsilon, \;\; \epsilon > 0,$$

we have :

$$V_i(x_i) \int_\Omega e^{u_i} dy - V(0) \int_\Omega e^u dy = o(1)$$

which means that there is no blow-up points.

Proof of the theorem

The Pohozaev identity gives us the following formula:

$$\int_\Omega < (x - x_2^i)|\nabla u_i > (-\Delta u_i) dy = \int_\Omega < (x - x_2^i)|\nabla u_i > V_i e^{u_i} dy = A_i$$

$$A_i = \int_{\partial B^+} < (x - x_2^i)|\nabla u_i >< \nu|\nabla u_i > d\sigma + \int_{\partial B^+} < (x - x_2^i)|\nu > |\nabla u_i|^2 d\sigma$$

We can write it as:

$$\int_\Omega < (x - x_2^i)|\nabla u_i > (V_i - V_i(x_i)) e^{u_i} dy = A_i + V_i(x_i) \int_\Omega < (x - x_2^i)|\nabla u_i > e^{u_i} dy =$$

$$= A_i + V_i(x_i) \int_\Omega < (x - x_2^i)|\nabla(e^{u_i}) > dy$$

And, if we integrate by part the second term, we have (because $x_1 = 0$ on the boundary and $\nu_2 = 0$):

$$\int_\Omega < (x - x_2^i)|\nabla u_i > (V_i - V_i(x_i)) e^{u_i} dy = -2V_i(x_i) \int_\Omega e^{u_i} dy + B_i$$

where B_i is,

$$B_i = V_i(x_i) \int_{\partial B^+} < (x - x_2^i)|\nu > e^{u_i} dy$$

applying the same procedure to u, we can write:

$$-2V_i(x_i) \int_\Omega e^{u_i} dy + 2V(0) \int_\Omega e^u dy = \int_\Omega < (x - x_2^i)|\nabla u_i > (V_i - V_i(x_i)) e^{u_i} dy -$$

$$-\int_{\Omega} <(x-x_2^i)|\nabla u>(V-V(0))e^u dy+(A_i-A)+(B_i-B),$$

where A and B are,

$$A=\int_{\partial B^+} <(x-x_2^i)|\nabla u><\nu|\nabla u>d\sigma+\int_{\partial B^+} <(x-x_2^i)|\nu>|\nabla u|^2 d\sigma$$

$$B=V_i(x_i)\int_{\partial B^+} <(x-x_2^i)|\nu>e^u dy$$

and, because of the uniform convergence of u_i and its derivatives on ∂B^+, we have:

$$A_i-A=o(1) \text{ and } B_i-B=o(1)$$

which we can write as:

$$V_i(x_i)\int_{\Omega} e^{u_i}dy-V(0)\int_{\Omega} e^u dy=\int_{\Omega} <(x-x_2^i)|\nabla(u_i-u)>(V_i-V_i(x_i))e^{u_i}dy+$$

$$+\int_{\Omega} <(x-x_2^i)|\nabla u>(V_i-V_i(x_i))(e^{u_i}-e^u)dy+$$

$$+\int_{\Omega} <(x-x_2^i)|\nabla u>(V_i-V_i(x_i)-(V-V(0)))e^u dy+o(1)$$

We can write the second term as:

$$\int_{\Omega} <(x-x_2^i)|\nabla u>(V_i-V_i(x_i))(e^{u_i}-e^u)dy=\int_{\Omega-B(0,\epsilon)} <(x-x_2^i)|\nabla u>(V_i-V_i(x_i))(e^{u_i}-e^u)dy+$$

$$+\int_{B(0,\epsilon)} <(x-x_2^i)|\nabla u>(V_i-V_i(x_i))(e^{u_i}-e^u)dy=o(1),$$

because of the uniform convergence of u_i to u outside a region which contain the blow-up and the uniform convergence of V_i. For the third integral we have the same result:

$$\int_{\Omega} <(x-x_2^i)|\nabla u>(V_i-V_i(x_i)-(V-V(0)))e^u dy=o(1),$$

because of the uniform convergence of V_i to V.

Now, we look to the first integral:

$$\int_{\Omega} <(x-x_2^i)|\nabla(u_i-u)>(V_i-V_i(x_i))e^{u_i}dy,$$

we can write it as:

$$\int_{\Omega} < (x-x_2^i)|\nabla(u_i-u) > (V_i-V_i(x_i))e^{u_i}dy = \int_{\Omega} < (x-x_i)|\nabla(u_i-u) > (V_i-V_i(x_i))e^{u_i}dy +$$

$$+\int_{\Omega} < x_1^i|\nabla(u_i-u) > (V_i-V_i(x_i))e^{u_i}dy,$$

Thus, we have proved by using the Pohozaev identity the following equality:

$$\int_{\Omega} < (x-x_i)|\nabla(u_i-u) > (V_i-V_i(x_i))e^{u_i}dy +$$

$$+\int_{\Omega} < x_1^i|\nabla(u_i-u) > (V_i-V_i(x_i))e^{u_i}dy =$$

$$= 2V_i(x_i)\int_{\Omega} e^{u_i}dy - 2V(0)\int_{\Omega} e^{u}dy + o(1)$$

it is sufficient to look to the integral on $B(x_i,\delta_i\epsilon)$, because u_i is bounded outside $B(x_i,\delta_i\epsilon)$ and the fact that :

$$||\nabla(u_i-u)||_1 = o(1).$$

Assume that we are in the case of one blow-up, it must be (x_i) and isolated, we can write the following inequality as a consequence of YY.Li-I.Shafrir result, in [8]:

$$u_i(x) + 2\log|x-x_i| \leq C,$$

We use this fact and the fact that V_i is s-holderian to have that, on $B(x_i,\delta_i\epsilon)$,

$$|(x-x_i)(V_i-V_i(x_i))e^{u_i}| \leq \frac{C}{|x-x_i|^{1-s}} \in L^{(2-\epsilon')/(1-s)},\ \forall\ \epsilon' > 0,$$

and, we use the fact that:

$$||\nabla(u_i-u)||_q = o(1),\ \forall\ 1 \leq q < 2$$

to conclude by the Holder inequality that:

$$\int_{B(x_i,\delta_i\epsilon)} < (x-x_2^i)|\nabla(u_i-u) > (V_i-V_i(x_i))e^{u_i}dy = o(1),$$

For the other integral, namely:

$$\int_{B(x_i,\delta_i\epsilon)} < x_1^i|\nabla(u_i-u) > (V_i-V_i(x_i))e^{u_i}dy,$$

We use the fact that, because our domain is a half ball, and the $\sup + \inf$ inequality to have:

$$x_1^i = \delta_i,\ u_i(x) + 4\log\delta_i \leq C$$

and,

$$e^{(s/2)u_i(x)} \leq |x - x_i|^{-s},\ |V_i - V_i(x_i)| \leq |x - x_i|^s,$$

Finaly, we have:

$$|\int_{B(x_i,\delta_i\epsilon)} < x_1^i|\nabla(u_i - u) > (V_i - V_i(x_i))e^{u_i}dy| \leq C\int_{B(x_i,\delta_i\epsilon)} |\nabla(u_i - u)|e^{((3/4)-(s/2))u_i},$$

But in the second member, for $1/2 < s \leq 1$, we have $q_s = 1/(3/4 - s/2) > 2$ and thus $q'_s < 2$ and,

$$e^{((3/4)-(s/2))u_i} \in L^{q_s},\ ||\nabla(u_i - u)||_{q'_s} = o(1),\ \forall\ 1 \leq q'_s < 2,$$

one conclude that:

$$\int_{B(x_i,\delta_i\epsilon)} < x_1^i|\nabla(u_i - u) > (V_i - V_i(x_i))e^{u_i}dy = o(1)$$

Finaly, with this method, we conclude that, in the case of one blow-up point and V_i is s-Holderian with $1/2 < s \leq 1$:

$$V_i(x_i)\int_\Omega e^{u_i}dy - V(0)\int_\Omega e^u dy = o(1)$$

which means that there is no blow-up, which is a contradiction.

Finaly, for one blow-up point and V_i is is s-Holderian with $1/2 < s \leq 1$, the sequence (u_i) is uniformly bounded on Ω.

The case of two blow-up points:

Theorem 2.4 *If V_i is s-Holderian with $1/2 < s \leq 1$ and,*

$$\int_\Omega V_i e^{u_i}dy \leq 24\pi - \epsilon,\ \epsilon > 0,$$

we have :

$$V_i(x_i)\int_\Omega e^{u_i}dy - V(0)\int_\Omega e^u dy = o(1)$$

which means that there is no blow-up points.

Proof of the Theorem

The case of two "interior" blow-up points:

As in the previous case, we assume that $\Omega = B^+$ is the half ball. We have two "interior" blow-up points x_i and y_i:

$$|y_i - x_i| \leq \delta_i \epsilon,$$

We use a Pohozaev type identity:

$$\int_{\Omega} < (x - x_2^i)|\nabla u_i > (-\Delta u_i) dy = \int_{\Omega} < (x - x_2^i)|\nabla u_i > V_i e^{u_i} dy = A_i$$

with A_i the regular part of the identity (on which the uniform convergence holds).

$$A_i = \int_{\partial B^+} < (x - x_2^i)|\nabla u_i >< \nu|\nabla u_i > d\sigma + \int_{\partial B^+} < (x - x_2^i)|\nu > |\nabla u_i|^2 d\sigma$$

We divide our domain in two domain Ω_1^i and Ω_2^i such that:

$$\Omega_1^i = \{x, |x - x_i| \leq |x - y_i|\},\ \Omega_2^i = \{x, |x - x_i| \geq |x - y_i|\}.$$

We set,

$$D_i = \{x, |x - x_i| = |x - y_i|\}.$$

We write:

$$A_i = \int_{\Omega_1^i} < (x - x_2^i)|\nabla u_i > (V_i - V_i(x_i)) e^{u_i} dy + \int_{\Omega_2^i} < (x - x_2^i)|\nabla u_i > (V_i - V_i(y_i)) e^{u_i} dy +$$

$$+ V_i(x_i) \int_{\Omega_1^i} < (x - x_2^i)|\nabla u_i > e^{u_i} dy + V_i(y_i) \int_{\Omega_2^i} < (x - x_2^i)|\nabla u_i > e^{u_i} dy.$$

As for the case of one blow-up point, it is sufficient to consider terms which contain the difference $\nabla(u_i - u)$.

We can write the last addition as (after using $\nabla(u_i - u)$) :

$$\left(V_i(x_i) \int_{\Omega} < (x - x_2^i)|\nabla u_i > e^{u_i} dy - \int_{\Omega} < (x - x_2^i)|\nabla u > e^{u} dy \right) +$$

$$+ (V_i(y_i) - V_i(x_i)) \int_{\Omega_2^i} < (x - x_2^i)|\nabla (u_i - u) > e^{u_i} dy.$$

First of all, we consider the term (which equal, after integration by part to):

$$V_i(x_i)\int_{\Omega} < (x-x_2^i)|\nabla u_i > e^{u_i}dy - \int_{\Omega} < (x-x_2^i)|\nabla u > e^{u}dy =$$

$$= -2V_i(x_i)\int_{\Omega} e^{u_i}dy + 2V(0)\int_{\Omega} e^{u}dy + (B_i - B)$$

with the same notation for B_i and B as for the previous case.

<u>Case 1:</u> suppose that, $|x-y_i| \geq |x_i - y_i|$,
thus

$$|V_i(x_i) - V_i(y_i)| \leq |x_i - y_i|^s \leq |x-y_i|^s$$

Thus,

$$|(V_i(y_i)-V_i(x_i))\int_{\Omega_2^i\cap\{x,|x-x_i|\geq|x-y_i|\}.} < (x-x_2^i)|\nabla(u_i-u) > e^{u_i}dy| \leq \int_{\Omega_2^i} |x-y_i|^{1+s}|\nabla(u_i-u)|e^{u_i}dy +$$

$$+|y_2^i - x_2^i|\int_{\Omega_2^i} |x-y_i|^s|\nabla(u_i-u)|e^{u_i}dy + |y_1^i|\int_{\Omega_2^i} |x-y_i|^s|\nabla(u_i-u)|e^{u_i}dy$$

But,

$$|y_i - x_i| \leq \delta_i\epsilon,\ x_1^i = \delta_i$$

we use the same method (with the $\sup + \inf$ inequality) to prove that for $1 \geq s > 1/2$ the two integrals converges to 0.

<u>Case 2:</u> suppose that, $|x-y_i| \leq |x_i - y_i|$,

We do integration by parts, we have one part on D_i and the other one on the circle with center y_i.

$$(V_i(y_i) - V_i(x_i))\int_{\Omega_2^i\cap\{x,|x-y_i|\leq|x_i-y_i|\}.} < (x-x_2^i)|\nabla(e^{u_i}) > dy =$$

$$= (V_i(y_i) - V_i(x_i))\int_{D_i\cap\{x,|x-y_i|\leq|x_i-y_i|\}.} < (x-x_2^i)|\nu > e^{u_i}dy +$$

$$+(V_i(y_i) - V_i(x_i))\int_{\{x,|x-y_i|=|x_i-y_i|\}\cap\{x,|x-y_i|\leq|x-x_i|\}} < (x-x_2^i)|\nu > e^{u_i}dy +$$

$$+2(V_i(y_i) - V_i(x_i))\int_{\{x,|x-y_i|\leq|x_i-y_i|\}} e^{u_i}dy$$

We set:

$$I_1 = (V_i(y_i) - V_i(x_i)) \int_{D_i \cap \{x, |x-y_i| \leq |x_i - y_i|\}.} < (x - x_2^i)|\nu > e^{u_i} dy,$$

$$I_2 = (V_i(y_i) - V_i(x_i)) \int_{\{x, |x-x_i| = |x_i - y_i|\} \cap \{x, |x-y_i| \leq |x - x_i|\}} < (x - x_2^i)|\nu > e^{u_i} dy$$

Lemma 2.5 *We have:*

$$I_1 = o(1), \;\; I_2 = o(1).$$

Proof of the lemma

For I_1, we have:

$$|V_i(x_i) - V_i(y_i)| \leq 2C|x - y_i|^s,$$

$$|I_1| \leq C \int_{D_i \cap \{x, |x-y_i| \leq |x_i - y_i|\}.} | < (x - y^i)|\nu > ||x - y_i|^s e^{u_i} +$$

$$+ |x_2^i - y_2^i| \int_{D_i \cap \{x, |x-y_i| \leq |x_i - y_i|\}.} |x - y_i|^s e^{u_i} dy +$$

$$+ |y_1^i| \int_{D_i \cap \{x, |x-y_i| \leq |x_i - y_i|\}.} |x - y_i|^s e^{u_i} dy$$

But,

$$x_1^i = \delta_i, \;\; |y_i - x_i| \leq \delta_i \epsilon,$$

$$u_i(x) + 4 \log \delta_i \leq C, \;\; e^{(3/4)u_i(x)} \leq |x - y_i|^{-3/2},$$

Thus,

$$|I_1| \leq \int_{D_i \cap \{x, |x-y_i| \leq |x_i - y_i|\}.} |x - y_i|^{s-1} +$$

$$+ C \int_{D_i \cap \{x, |x-y_i| \leq |x_i - y_i|\}.} |x - y_i|^{(-3/2)+s} dy,$$

If we set $t_0 = (x_i + y_i)/2$, we have on one part of D_i:

$$|x - t_0| \leq |x - y_i| = |x - x_i| \leq |x_i - y_i|,$$

by a change of variable $u = x - t_0$ on the line D_i, we can compute the two last integrals directly, to have, for $1 \geq s > 1/2$:

$$|I_1| \leq C(|x_i-y_i|^s+|x_i-y_i|^{s-(1/2)})=o(1),$$

For I_2 we have:

$$I_2=(V_i(y_i)-V_i(x_i))\int_{\{x,|x-y_i|=|x_i-y_i|\}\cap\{x,|x-x_i|\leq|x_i-y_i|\}} <(x-x_2^i)|\nu> e^{u_i}dy$$

and,

$$|V_i(x_i)-V_i(y_i)|\leq 2C|x-y_i|^s,$$

$$|I_2|\leq C\int_{\{x,|x-y_i|=|x_i-y_i|\}\cap\{x,|x-y_i|\leq|x-x_i|\}.} |<(x-y_i)|\nu>||x-y_i|^s e^{u_i}+$$

$$+|x_2^i-y_2^i|\int_{\{x,|x-y_i|=|x_i-y_i|\}\cap\{x,|x-y_i|\leq|x-x_i|\}.}|x-y_i|^s e^{u_i}dy+$$

$$+|y_1^i|\int_{\{x,|x-y_i|=|x_i-y_i|\}\cap\{x,|x-x_i|\leq|x-x_i|\}.}|x-y_i|^s e^{u_i}dy$$

with the same method as for I_1 we have:

$$|I_2|\leq C\int_{\{x,|x-y_i|=|x_i-y_i|\}\cap\{x,|x-y_i|\leq|x-x_i|\}.}|x-y_i|^{s-1}+$$

$$+\int_{\{x,|x-y_i|=|x_i-y_i|\}\cap\{x,|x-y_i|\leq|x-x_i|\}.}|x-y_i|^{-(3/2)+s}dy,$$

Finaly, we have:

$$|I_2| \leq C(|x_i-y_i|^s+|x_i-y_i|^{s-(1/2)})=o(1),$$

<u>The case of two "exteriror" blow-up points:</u>

Let $(x_i)_i$ and $(t_i)_i$ two sequences of "exterior" blow-up points. If $d(x_i,t_i)=O(\delta_i)$ or $d(x_i,t_i)=O(\delta_i')$ then we use the same technique as for two interior blow-up with the Pohozaev identity. In this case the $\sup+\inf$ inequality holds, because $d(x_i,t_i)$ is of order δ_i or δ_i'. Assume that:

$$\frac{d(x_i,t_i)}{\delta_i}\to+\infty \ \text{ and } \ \frac{d(x_i,t_i)}{\delta_i'}\to+\infty$$

In this case, we assume that, we are on the half ball. By a conformal transformation, f, we can assume that our two sequences are on the unit ball. First of all, we use the Pohozaev identity on the half ball as for the previous cases, but our domain change, we have one part is vertical, the second part is a part of the boundary of the unit ball, in which the sequences (u_i) and $(\partial u_i)_i$ are uniformly bounded and converge to the corresponding function, and the third part of boundary, is a regular curve D_i' such that its image by f is the mediatrice D_i of the segment (x_i, t_i). In the Pohozaev identity, we have a terms of type:

$$\int_{D_i'} < (x - x_2^i)|\nabla u_i >< \nu|\nabla u_i > d\sigma + \int_{D_i'} < (x - x_2^i)|\nu > |\nabla u_i|^2 d\sigma$$

But if we integrate on the rest of the domain and if we use the Pohozeav identity on this second domain and we replace x_2^i by t_2^i, the integral on D_i' is :

$$-\int_{D_i'} < (x - t_2^i)|\nabla u_i >< \nu|\nabla u_i > d\sigma - \int_{D_i'} < (x - t_2^i)|\nu > |\nabla u_i|^2 d\sigma$$

If, we add the two integral, we find:

$$\int_{D_i'} < (x_2^i - t_2^i)|\nabla u_i >< \nu|\nabla u_i > d\sigma + \int_{D_i'} < (x_2^i - t_2^i)|\nu > |\nabla u_i|^2 d\sigma$$

We have the same techniques as for the previous cases ("interior" blow-up), except the fact that here, we use the Pohozaev identity on two differents domains which the union is our half ball.

To conclude, we must show that this last integral is close to 0 as i tends to $+\infty$. By a conformal change of the metric, it is sufficient to prove that the corresponding integral on the unit ball on D_i tends to 0. Without loss of generality, we can assume here that we work on the unit ball (for this integral).

On the unit ball, with the Dirichlet condition, the Green function is (in complex notation) :

$$G(x,y) = \frac{1}{2\pi}\log\frac{|1-\bar{x}y|}{|x-y|},\ u_i(x) = \int_{B_1(0)} G(x,y)V_i(y)e^{u_i(y)}dy,$$

We can compute (in complex notation) $\partial_x G$ and $\partial_x u_i$:

$$\partial_x G(x,y) = \frac{1-|y|^2}{(x-y)(x\bar{y}-1)},$$

$$\partial_x u_i(x) = \int_{B_1(0)} \partial_x G(x,y)V_i(y)e^{u_i(y)}dy = \int_{B_1(0)} \frac{1-|y|^2}{(x-y)(x\bar{y}-1)}V_i(y)e^{u_i(y)}dy$$

Let $t_0^i = (x_i + t_i)/2$. We assume that $|x - t_0^i| \leq 1 - \epsilon$ and $|t_0^i| \geq 1 - (\epsilon/2)$.

Proposition 2.6 *1) For* $((1/2)+\tilde{\epsilon})|x_i-t_i| \leq |x-t_0^i| \leq 1-\epsilon$ *we have,*

$$|\partial_x u_i(x)| \leq C' + C\frac{\delta_i}{|x_i-t_i|}\frac{1}{|x-t_0^i|} = C' + \frac{o(1)}{|x-t_0^i|}.$$

2) For $|x-t_0^i| \leq ((1/2)-\tilde{\epsilon})|x_i-t_i|$ *we have,*

$$|\partial_x u_i(x)| \leq C' + C\frac{\delta_i}{|x_i-t_i|}\frac{1}{|x_i-t_0^i|} = C' + \frac{o(1)}{|x_i-t_0^i|}.$$

with $o(1) \to 0$ *as* $i \to +\infty$ *.*

3) For $((1/2)-\tilde{\epsilon})|x_i-t_0^i| \leq |x-t_0^i| \leq ((1/2)+\tilde{\epsilon})|x_i-t_i|$ *we have,*

$$|x_i-t_i||\nabla u_i|_{L^\infty(D_i \cap \{((1/2)-\tilde{\epsilon})|x_i-t_0^i| \leq |x-t_0^i| \leq ((1/2)+\tilde{\epsilon})|x_i-t_i|\}} \leq C.$$

Proof of the proposition:

To estimate $\partial_x u_i$ on D_i, we divide the last integral in three parts:

$$\partial_x u_i(x) = \int_{B_1(0)-(B(x_i,\delta_i\epsilon)\cup B(t_i,\delta_i'\epsilon))} \frac{1-|y|^2}{(x-y)(x\bar{y}-1)} V_i(y)e^{u_i(y)}dy +$$

$$+ \int_{B(x_i,\delta_i\epsilon)} \frac{1-|y|^2}{(x-y)(x\bar{y}-1)} V_i(y)e^{u_i(y)}dy +$$

$$+ \int_{B(t_i,\delta_i'\epsilon)} \frac{1-|y|^2}{(x-y)(x\bar{y}-1)} V_i(y)e^{u_i(y)}dy$$

Let us set:

$$I_1 = \int_{B_1(0)-(B(x_i,\delta_i\epsilon)\cup B(t_i,\delta_i'\epsilon))} \frac{1-|y|^2}{(x-y)(x\bar{y}-1)} V_i(y)e^{u_i(y)}dy$$

$$I_2 = \int_{B(x_i,\delta_i\epsilon)} \frac{1-|y|^2}{(x-y)(x\bar{y}-1)} V_i(y)e^{u_i(y)}dy,$$

$$I_3 = \int_{B(t_i,\delta_i'\epsilon)} \frac{1-|y|^2}{(x-y)(x\bar{y}-1)} V_i(y)e^{u_i(y)}dy$$

For the first integral, because $u_i \leq C$ on $B_1(0)-(B(x_i,\delta_i\epsilon)\cup B(t_i,\delta_i'\epsilon))$, we have:

$$|I_1| \leq C\int_{B_1(0)} \frac{1-|y|^2}{|x-y||x\bar{y}-1|}dy,$$

But, $1 \geq |x| = |x - t_0^i + t_0^i| \geq |t_0^i| - |x - t_0^i| \geq 1 - (\epsilon/2) - (1-\epsilon) = \epsilon/2$, thus, we can write:

$$|I_1| \leq C \int_{B_1(0)} \frac{1-|y|^2}{|x-y||x||\bar{y}-1/x|} dy,$$

and, we use the fact that:

$$|\bar{y} - 1/x| \geq ||\bar{y}| - 1/|x|| \geq |1/|x| - |y|| \geq (1-|y|),$$

To have:

$$|\partial_x u_i(x)| \leq |I_2| + |I_3| + C \int_{B_1(0)} \frac{1+|y|}{|x-y|} dy = |I_2| + |I_3| + C',$$

Now, we look to the second and third integrals, it is sufficient to consider the first one :

$$I_2 = \int_{B(x_i, \delta_i \epsilon)} \frac{1-|y|^2}{(x-y)(x\bar{y}-1)} V_i(y) e^{u_i(y)} dy$$

<u>Case 1: $((1/2)+\tilde{\epsilon})|x_i - t_i| < |x - t_0^i| < 1 - \epsilon$:</u>

In this case we have:

$$1 - |y|^2 = 1 - |x_i + \delta_i z|^2 = \delta_i (2 + o(1)),$$

and,

$$|x - y| = |x - t_0^i + t_0^i - y_i - \delta_i z| \geq (\tilde{\epsilon}/2)|x_i - t_i|,$$

and,

$$|x\bar{y} - 1| = |((x - t_0^i + t_0^i - x_i) + x_i)(\bar{x}_i + \delta_i \bar{z}) - 1| \geq (\tilde{\epsilon}/2)|x - t_0^i|,$$

Thus,

$$|\partial_x u_i(x)| \leq C' + C \frac{\delta_i}{|x_i - t_i|} \frac{1}{|x - t_0^i|} = C' + \frac{o(1)}{|x - t_0^i|}.$$

with, $o(1) \to 0$ as $i \to +\infty$.

<u>Case 2: $|x - t_0^i| < ((1/2) - \tilde{\epsilon})|x_i - t_i|$:</u>

In this case, we have:

$$1 - |y|^2 = 1 - |x_i + \delta_i z|^2 = \delta_i (2 + o(1)),$$

and,

$$|x-y| = |x - t_0^i + t_0^i - y_i - \delta_i z| \geq (\tilde{\epsilon}/2)|x_i - t_i|,$$

and,

$$|x\bar{y} - 1| = |((x - t_0^i + t_0^i - x_i) + x_i)(\bar{x}_i + \delta_i \bar{z}) - 1| \geq (\tilde{\epsilon}/2)|x_i - t_0^i|,$$

Thus,

$$|\partial_x u_i(x)| \leq C' + C\frac{\delta_i}{|x_i - t_i|}\frac{1}{|x_i - t_0^i|} = C' + \frac{o(1)}{|x_i - t_0^i|}.$$

with, $o(1) \to 0$ as $i \to +\infty$.

Case 3: $((1/2) - \tilde{\epsilon})|x_i - t_0^i| < |x - t_0^i| < ((1/2) + \tilde{\epsilon})|x_i - t_i|$:

Let $\tilde{t}_0^i$ the point of D_i such that $|\tilde{t}_0^i - t_0^i| = 1/2(|x_i - t_0^i|)$. We use the fact that the function:

$$v_i(t) = u_i(\tilde{t}_0^i + (|x_i - t_0^i|/4)t),$$

is uniformly bounded for $|t| \leq 1$ and is a solution of PDE which is uniformly bounded on $|t| \leq 1$. By the elliptic estimates we have:

$$|x_i - t_i||\nabla u_i|_{L^\infty(D_i \cap \{((1/2)-\tilde{\epsilon})|x_i - t_0^i| \leq |x - t_0^i| \leq ((1/2)+\tilde{\epsilon})|x_i - t_i|\}} \leq C.$$

Thus, we use the previous cases to compute the following integral:

$$\int_{D_i} < (x_2^i - t_2^i)|\nabla u_i >< \nu|\nabla u_i > d\sigma + \int_{D_i} < (x_2^i - t_2^i)|\nu > |\nabla u_i|^2 d\sigma = o(1)$$

and, thus we have the same estimate for the integral on D_i'.
here, we used the previous estimates with $i \to +\infty$ and $\tilde{\epsilon} \to 0$ (for the previous case 3).

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