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# Some Properties of Distributional Wedge Products

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**Abstract**

This paper considers the distributional wedge product. Some properties are proved, which can be used in the study of quasiregular mappings and mappings of finite distortion.

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# 1 Introduction

In the theory of non-linear differential forms and their applications to modern theory of mappings, one of the most important concepts is the distributional wedge product. As a generalization of distributional Jacibian, it has important applications in the theory of geometric function theory and non-linear analysis, see [1-3]. In this paper, we give some properties of the distributional wedge products.

Let $f = (f^1, f^2, \cdots, f^n) : \Omega \to \mathrm{R}^n$ be a Sobolev mapping. Given a pair of ordered $\ell$-tuples $I = (i_1, i_2, \cdots, i_\ell)$ and $J = (j_1, j_2, \cdots, j_\ell)$, there is an associated $\ell \times \ell$ minor of the differential matrix $Df = \left(\frac{\partial f^i}{\partial x_j}\right)_{1\le i,j\le n}$. We shall use the following notation for such minors

$$\frac{\partial^I f}{\partial x_J} = \frac{\partial(f^{i_1}, f^{i_2}, \cdots, f^{i_\ell})}{\partial(x_{j_1}, x_{j_2}, \cdots, x_{j_\ell})} = \det\left[\frac{\partial f^i}{\partial x_j}\right]_{i\in I, j\in J}.$$

Thus the $(i,j)$th entry of $Df$ is obtained when $I = (i)$ and $J = (j)$, while the Jacobian determinant is obtained when $I = J = N = (1, 2, \cdots, n)$. For $J = (j_1, j_2, \cdots, j_\ell)$, denote by $N - J = (k_1, k_2, \cdots, k_{n-\ell})$ obtained from $N = (1, 2, \cdots, n)$ by deleting all terms in $J$.

Let $e_1, e_2, \cdots, e_n$ denote the standard basis of $\mathrm{R}^n$. For each $\ell = 0, 1, \cdots, n$ denote by $\bigwedge^\ell = \bigwedge^\ell(\mathrm{R}^n)$ the space of $\ell$-covectors on $\mathrm{R}^n$, $\bigwedge^0 = \mathrm{R}$, $\bigwedge^1 = \mathrm{R}^n$. Then $\bigwedge^\ell$ consists of linear combinations of exterior products

$$e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell},$$

where $I = (i_1, i_2, \cdots, i_\ell)$ is an $\ell$-tuple.

For a smooth mapping $f = (f^1, f^2, \cdots, f^n) : \Omega \to \mathrm{R}^n$ and $1 \le l \le n$, one can use Stoke's theorem to write

$$\int_\Omega \varphi(x) df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J} = -\int_\Omega f^{i_1} d\varphi \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J},$$

where $\varphi \in C_0^\infty(\Omega)$. This later integral actually converges for mappings in the Sobolev space $W^{1,s}_{loc}(\Omega, \mathrm{R}^n)$, with $s = \frac{nl}{n+1}$. Indeed, we have

$$\left| d\varphi \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \right| \le |\nabla\varphi||Df|^{l-1},$$

and this last term lies in $L_{loc}^{\frac{n\ell}{(n+1)(\ell-1)}}(\Omega)$, whereas $f^{i_1}$ is locally in the dual space $L_{loc}^{\frac{n\ell}{n-\ell+1}}(\Omega)$, by the Sobolev embedding theorem. An immediate consequence of this is that we are able to make the following definition.

**Definition 1.1** *The distributional wedge product is defined for mappings* $f \in W^{1,s}_{loc}(\Omega, R^n)$ *with* $s = \frac{n\ell}{n+1}$ *and any ordered* $\ell$*-tuples* $I = (i_1, \cdots, i_\ell)$ *and* $J = (j_1, \cdots, j_\ell)$ *by the rule*

$$\mathcal{J}^J_{f^I}[\varphi] = -\int_\Omega f^{i_1} d\varphi \wedge df^{i_2} \wedge \cdots \wedge df^{i_\ell} \wedge dx_{N-J}$$

*for* $\varphi \in C_0^\infty(\Omega)$.

This definition gives us the continuous non-linear operator

$$\mathcal{J}^J_{f^I} : W_{loc}^{1,\frac{n\ell}{n+1}}(\Omega, \mathrm{R}^n) \to \mathcal{D}'(\Omega),$$

where $\mathcal{D}'(\Omega)$ represents the dual space to $C_0^\infty(\Omega)$, that is, the space of Schwarz distributions. If $\ell = n$, then the distributional wedge product coincides with the distributional Jacobian, see [1].

## 2 Some Properties of Distributional Wedge Products

In the following, $C(n)$ is some constant depending only on the dimension $n$, it may vary from line to line. In this section, we give some properties of distributional wedge products. The first result to consider is the following theorem.

**Theorem 2.1** *Let $f \in W^{1,s}_{loc}(\Omega, R^n)$, $s = \frac{n\ell}{n+1}$, and $Q \subset \Omega$ be a cube. If the test function $\varphi \in C_0^\infty(Q)$ satisfies $|\nabla\varphi| \le C(n)/diam(Q)$, then for any ordered $\ell$-tuples $I = (i_1, \cdots, i_\ell)$ and $J = (j_1, \cdots, j_\ell)$, we have*

$$\left|\mathcal{J}^J_{f^I}[\varphi]\right| = \left|\int_Q \varphi(x) df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J}\right| \le C(n)\,|Q|^{1-\frac{l}{s}}\Big(\int_Q |Df|^s\Big)^{\frac{l}{s}}.$$

**Proof.** We only need to prove the last inequality. Denote by $f^{i_1}_Q$ the integral mean of $f^{i_1}$ over $Q$, that is, $f^{i_1}_Q = ⨍_Q f^{i_1} dx$. By Stoke's theorem, Hölder's inequality and Poincaré-Sobolev inequality, we have

$$\begin{aligned}
&\left|\mathcal{J}^J_{f^I}[\varphi]\right| = \left|\int_Q \varphi(x) df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J}\right| \\
&= \left|\int_Q (f^{i_1} - f^{i_1}_Q) d\varphi \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J}\right| \\
&\le \int_Q |f^{i_1} - f^{i_1}_Q||\bigtriangledown \varphi||Df|^{\ell-1} dx \le \frac{C(n)}{\text{diam (Q)}} \int_Q |f^{i_1} - f^{i_1}_Q||Df|^{\ell-1} dx \\
&\le \frac{C(n)}{\text{diam (Q)}} \left(\int_Q |f^{i_1} - f^{i_1}_Q|^{\frac{n\ell}{n-\ell+1}} dx\right)^{\frac{n-\ell+1}{n\ell}} \left(\int_Q |Df|^{\frac{n\ell}{n+1}} dx\right)^{\frac{(n+1)(\ell-1)}{n\ell}} \\
&\le \frac{C(n)}{\text{diam (Q)}} \left(\int_Q |Df|^{\frac{n\ell}{n+1}} dx\right)^{\frac{n+1}{n}} = C(n)\,|Q|^{1-\frac{l}{s}} \left(\int_Q |Df|^s\right)^{\frac{l}{s}}.
\end{aligned}$$

This ends the proof of Theorem 2.1.

**Theorem 2.2** *If $n-l+1$ coordinate functions of a mapping $f = (f^1, f^2, \cdots, f^n) \in W^{1,\ell}(\Omega, R^n)$ vanish on $\partial\Omega$ in the Sobolev sense, then for any ordered $\ell$-tuples $I = (i_1, \cdots, i_\ell)$ and $J = (j_1, \cdots, j_\ell)$,*

$$\int_\Omega df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J} = 0. \tag{2.1}$$

*If two mappings $f, g \in W^{1,\ell}(\Omega, R^n)$ agree on $\partial\Omega$ in the Sobolev sense, then*

$$\int_\Omega df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_\ell} \wedge dx_{N-J} = \int_\Omega dg^{i_1} \wedge dg^{i_2} \wedge \cdots \wedge dg^{i_\ell} \wedge dx_{N-J}. \tag{2.2}$$

**Proof.** It is no loss of generality to assume that $f^{i_k}$ vanishes on $\partial\Omega$ in the Sobolev sense, for some $k \in \{1, 2, \cdots, \ell\}$, because otherwise we will have a contradiction. Stoke's theorem yields

$$\begin{aligned} & \int_\Omega df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J} \\ = \quad & (-1)^{k-1} \int_\Omega d\left(f^{i_k} df^{i_1} \wedge \cdots \wedge \widehat{df^{i_k}} \wedge \cdots \wedge df^{i_\ell} \wedge dx_{N-J}\right) \\ = \quad & (-1)^{k-1} \int_{\partial\Omega} f^{i_k} df^{i_1} \wedge \cdots \wedge \widehat{df^{i_k}} \wedge \cdots \wedge df^{i_\ell} \wedge dx_{N-J} = 0 \end{aligned}$$

where the circumflex over a term means it is to be omitted. If $f, g \in W^{1,\ell}(\Omega, \mathrm{R}^n)$ agree on $\partial\Omega$ in the Sobolev sense, then by (2.1),

$$\begin{aligned} & \int_\Omega df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_\ell} \wedge dx_{N-J} - \int_\Omega dg^{i_1} \wedge dg^{i_2} \wedge \cdots \wedge dg^{i_\ell} \wedge dx_{N-J} \\ = & \sum_{k=1}^{\ell} \int_\Omega df^{i_1} \wedge \cdots \wedge d(f^{i_k} - g^{i_k}) \wedge \cdots \wedge dg^{i_\ell} \wedge dx_{N-J} = 0. \end{aligned}$$

From this (2.2) follows. This ends the proof of Theorem 2.2.

We now need a few facts from harmonic analysis in order to state and prove Theorem 2.3. For $h \in L^s(\mathrm{R}^n), 1 \leq s < \infty$, the maximal function of $h$ is defined by

$$(M_s h)(x) = \sup \left\{ \left( \frac{1}{|Q|} \int_Q |h|^s \right)^{\frac{1}{s}} : x \in Q \subset \mathrm{R}^n \right\}.$$

The following result represents a slight strengthening of the well-known weak-type inequality

$$|\{x : M_s h(x) > 2t\}| \leq \frac{C(n,s)}{t^s} \int_{|h(x)|>t} |h(x)|^s dx.$$

Another prerequisite for the proof of Theorem 3 is the Whitney decomposition and the adjusted partition of unity, see [4]. Let $F$ be a non-empty closed set in $\mathrm{R}^n$ and $\Omega$ its complement. Then there is a collection $\mathcal{F} = \{Q_1, Q_2, \cdots\}$ of non-overlapping cubes such that

1. $\Omega = \bigcup_{i=1}^\infty Q_i$;

2. diam $Q_i \leq$ dist $(Q_i, F) \leq 4\mathrm{diam} Q_i$;

3. $\lambda Q_i$ intersects $F$ if $\lambda \geq 7n$.

Here we denote by $\lambda Q$ the cube which has the same centre as $Q$ but is expanded (or contracted) by the factor $\lambda$. The last fact follows from elementary geometric considerations. We follow the notation used in [4] and write $Q_i^* = \frac{11}{10} Q_i$. Now there exists a partition of unity $1 = \sum_{i=1}^{\infty} \varphi_i(x)$, $x \in \Omega$, where $\varphi_i \in C_0^\infty(Q_i^*)$ are non-negative functions such that

$$| \nabla \varphi_i(x) | \leq \frac{C(n)}{\mathrm{diam}(Q_i)}, \quad i = 1, 2, \cdots$$

**Theorem 2.3** *Let $f \in W^{1,s}(R^n, R^n)$ with $s = \frac{nl}{n+1}$ have compact support and let*

$$M(x) = (M_s|Df|)(x).$$

*Then for all but a countable number of $t > 0$ we have*

$$\left| \int_{M(x)\leq 2t} df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J} \right| \leq C(n) t^{l-s} \int_{|Df(x)|>t} |Df(x)|^s dx. \quad (2.3)$$

An Orlicz function is a continuously increasing function satisfying

$$P : [0, \infty) \to [0, \infty), \ P(0) = 0 \text{ and } \lim_{t\to\infty} P(t) = \infty.$$

The Orlicz space $L^P(\Omega, \mathrm{R}^n)$ consists of those Lebesgue measurable mappings defined in $\Omega$ and valued in the space $\mathrm{R}^n$ such that

$$\int_\Omega P(\lambda|f|)dx < \infty \text{ for some } \lambda = \lambda(f) > 0.$$

The spaces $L^P_{loc}(\Omega, \mathrm{R}^n)$ and $W^{1,P}_{loc}(\Omega, \mathrm{R}^n)$ are easy to understand.

**Theorem 2.4** *Under the divergence condition*

$$\int_1^\infty P(t) \frac{dt}{t^{l+1}} = +\infty \quad (2.4)$$

*and the convexity assumption*

$$t \to P(t^{\frac{n+1}{nl}}) \text{ is convex} \quad (2.5)$$

*on the Orlicz function $P = P(t)$, we have for each $H \in L^P(R^n)$,*

$$\liminf_{t\to\infty} t^{l-s} \int_{|H(x)|>t} |H(x)|^s dx = 0,$$

*where $s = \frac{nl}{n+1}$.*

The proof of Theorems 2.3 and 2.4 are just a little modifications of [1, Lemmas 7.2.1 and 7.2.2] by using the maximal function, Whitney decomposition and Theorem 2.2. We omit the details.

**Theorem 2.5** *Let $f$ lies in $W^{1,P}_{loc}(\Omega, R^n)$ with the Orlicz function $P$ satisfying the divergen condition (2.4) and the convexity condition (2.5). Then for any ordered $\ell$-tuples $I = (i_1, \cdots, i_\ell)$ and $J = (j_1, \cdots, j_\ell)$, if*

$$\frac{\partial^I f}{\partial x_J} = \frac{\partial(f^{i_1}, f^{i_2}, \cdots, f^{i_l})}{\partial(x_{j_1}, x_{j_2}, \cdots, x_{j_l})} \geq 0,$$

*then it is locally integrable, and*

$$\mathcal{J}^J_{f^I}[\varphi] = \int_\Omega \varphi(x) df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J} = (-1)^{\sigma(J,N-J)} \int_\Omega \varphi(x) \frac{\partial f^I}{\partial x_J} dx$$

*for every test function $\varphi \in C_0^\infty(\Omega)$, where $\sigma(J, N-J)$ is the sign of the induced permutation which is either odd or even.*

**Proof** We choose an arbitrary non-negative test function $\varphi \in C_0^\infty(\Omega)$. We choose yet another test function $\eta \in C_0^\infty(\Omega)$ which is equal to 1 on the support of $\varphi$. Thus

$$\frac{\partial(\varphi f^{i_1}, f^{i_2}, \cdots, f^{i_l})}{\partial(x_{j_1}, x_{j_2}, \cdots, x_{j_l})} = \frac{\partial(\varphi f^{i_1}, \eta f^{i_2}, \cdots, \eta f^{i_l})}{\partial(x_{j_1}, x_{j_2}, \cdots, x_{j_l})}.$$

Note that the mapping $f' = (\varphi f^1, \eta f^2, \cdots, \eta f^n)$ lies in the Orlicz-Sobolev space $W^{1,P}(\mathrm{R}^n, \mathrm{R}^n)$. Let

$$M'(x) = (M_s|Df'|)(x).$$

Because of Theorems 2.3 and 2.4, we have

$$\liminf_{t\to\infty} \left| \int_{M'(x)<2t} d(\varphi f^{i_1}) \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J} \right| = 0. \tag{2.6}$$

We now split the integrand as

$$\begin{aligned} & d(\varphi f^{i_1}) \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J} \\ = \ & \varphi(x) df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J} + f^{i_1} d\varphi \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J}. \end{aligned}$$

The second term here is in fact integrable on $\Omega$ and, by the very definition of distributional wedge product, we have

$$\lim_{t\to\infty} \int_{M'(x)<2t} f^{i_1} d\varphi \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J} = -\mathcal{J}^J_{f^I}[\varphi].$$

The first term is non-negative, so the limit of the integral in question exists and is equal to $\mathcal{J}^J_{f^I}[\varphi]$. That is

$$\lim_{t\to\infty} \int_{M'<2t} \varphi(x) df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J} = \mathcal{J}^J_{f^I}[\varphi]$$

by (2.6). Now the monotone convergence theorem makes it possible for us to pass to the limit under the domain of integration to obtain

$$\int_\Omega \varphi(x) df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J} = \mathcal{J}^J_{f^I}[\varphi] \tag{2.7}$$

for every non-negative test function $\varphi \in C_0^\infty(\Omega)$. This shows, in particular, that the

$$\frac{\partial^I f}{\partial x_J} = \frac{\partial(f^{i_1}, f^{i_2}, \cdots, f^{i_l})}{\partial(x_{j_1}, x_{j_2}, \cdots, x_{j_l})}$$

is locally integrable. To complete the proof we note that we did not really have to use a smooth test function $\varphi \in C_0^\infty(\Omega)$. The same arguments work for non-negative Lipschitz functions. If $\varphi \in C_0^\infty(\Omega)$ changes sign, we can apply (2.7) to the positive and negative parts of $\varphi$ respectively. The identity at (2.7) remains valid for all test functions, completing the proof of Theorem 2,5.

**Theorem 2.6** *Under the same conditions with Theorem 5, if $n - l + 1$ coordinate functions of $f = (f^1, f^2, \cdots, f^n)$ lies in $W_0^{1,P}(B)$ for some relatively compact subdomain $B \subset \Omega$, then for any ordered $\ell$-tuples $I = (i_1, \cdots, i_\ell)$ and $J = (j_1, \cdots, j_\ell)$,*

$$\frac{\partial^I f}{\partial x_J} = \frac{\partial(f^{i_1}, f^{i_2}, \cdots, f^{i_l})}{\partial(x_{j_1}, x_{j_2}, \cdots, x_{j_l})} \equiv 0$$

*almost everywhere in $B$.*

**Proof** It is no loss of generality to assume $f^{i_1} \in W_0^{1,P}(B)$. Then its extension by zero lies in $W^{1,P}(\Omega)$ and we have

$$\frac{\partial(f^{i_1}, f^{i_2}, \cdots, f^{i_l})}{\partial(x_{j_1}, x_{j_2}, \cdots, x_{j_l})} = 0$$

outside $B$. Consider a test function $\varphi \in C_0^\infty(\Omega)$ equal to 1 on $B$. Then

$$\begin{aligned} & \int_B df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J} \\ = & \int_\Omega \varphi(x) df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J} = \mathcal{J}^J_{f^I}[\varphi] \\ = & -\int_\Omega f^{i_1} d\varphi \wedge df^{i_2} \wedge \cdots \wedge df^{i_l} \wedge dx_{N-J} = 0. \end{aligned}$$

Since we have $\frac{\partial^I f}{\partial x_J} \geq 0$ almost everywhere, it follows that

$$\frac{\partial(f^{i_1}, f^{i_2}, \cdots, f^{i_l})}{\partial(x_{j_1}, x_{j_2}, \cdots, x_{j_l})} = 0$$

almost everywhere.

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# References

[1] T.Iwaniec, G.Martin, Geometric function theory and non-linear analysis, Clarendon Press, Oxford, 2001.

[2] T.Iwaniec, G.Martin, Quasiregular mappings in even dimensions, Acta Math., 1993, 170: 29-81.

[3] T.Iwaniec, $p$-harmonic tensors and quasiregular mappings, Ann. of Math., 1992, 136: 589-624.

[4] E.M.Stein, Some results in harmonic analysis in $\mathrm{R}^n$ for $n \to \infty$, Bull. Amer. Math. Soc., 1983, 9: 71-73.

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