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Coclosed-Exact Fields of Differential Forms

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Abstract

In the present paper, we first give the definition for coclosed-exact fields of differential forms, and then an estimate below the natural exponents of coclosed-exact forms is obtained. An application to the regularity theory of quasiregular mappings is given.

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1 Introduction

We first introduce some basic notions of exterior calculus. Throughout this paper we always assume Ω is a connected open subset of R^n, $n \geq 2$. We use $e_1, e_2, \cdots, e_n$ to denote the standard unit basis of R^n. Let $\bigwedge^\ell = \bigwedge^\ell(\mathrm{R}^n)$ be the linear space of ℓ-covectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell}$, corresponding to all ordered ℓ-tuples $I = (i_1, i_2, \cdots, i_\ell)$, $1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$, $\ell = 0, 1, \cdots, n$. The Grassman algebra $\bigwedge = \oplus \bigwedge^\ell$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \bigwedge$ and $\beta = \sum \beta^I e_I \in \bigwedge$, the inner product in $\bigwedge$ is given by $\langle \alpha, \beta \rangle = \sum_I \alpha^I \beta^I$ with summation over all ℓ-tuples $I = (i_1, i_2, \cdots, i_\ell)$ and all integers $\ell = 0, 1, \cdots, n$. The Hodge star operator $* : \bigwedge \to \bigwedge$ is defined by the rule $*1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle (*1)$ for all $\alpha, \beta \in \bigwedge$. The norm of $\alpha \in \bigwedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = *(\alpha \wedge *\alpha) \in \bigwedge^0 = \mathrm{R}$. The Hodge star is an isometric isomorphism on $\bigwedge$ with $* : \bigwedge^\ell \to \bigwedge^{n-\ell}$ and $** = (-1)^{\ell(n-\ell)} : \bigwedge^\ell \to \bigwedge^\ell$.

Let $\mathcal{D}'\left(\Omega, \bigwedge^\ell\right)$ be those differential forms $\omega = \sum \omega^I e_I \in \bigwedge^\ell$ with $\omega^I \in \mathcal{D}'(\Omega)$, where we have denoted by $\mathcal{D}'(\Omega)$ the space of Schwartz distributions. Let $1 \leq p < \infty$. We denote the L^p-norm of a measurable function f over Ω by

$$\|f\|_p = \|f\|_{p,\Omega} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}.$$

We write $L^p\left(\Omega,\bigwedge^\ell\right)$ for the ℓ-forms $\omega(x)=\sum_I\omega_I(x)dx_I=\sum\omega_{i_1i_2\cdots i_\ell}(x)dx_{i_1}\wedge dx_{i_2}\wedge\cdots\wedge dx_{i_\ell}$ with $\omega_I(x)\in L^p(\Omega,\mathrm{R})$ for all ordered ℓ-tuples I. Thus $L^p\left(\Omega,\bigwedge^\ell\right)$ is a Banach space with norm

$$\|\omega\|_{p,\Omega}=\left(\int_\Omega|\omega(x)|^pdx\right)^{1/p}=\left(\int_\Omega\left(\sum|\omega_I(x)|^2\right)^{p/2}dx\right)^{1/p}.$$

Similarly, $W^{1,p}\left(\Omega,\bigwedge^\ell\right)$ are those differential ℓ-forms on Ω whose coefficients are in $W^{1,p}(\Omega,\mathrm{R})$. The notations $W^{1,p}_{loc}(\Omega,\mathrm{R})^n$ and $W^{1,p}_{loc}\left(\Omega,\bigwedge^\ell\right)$ are self-explanatory. The exterior derivative is denoted by $d:\mathcal{D}'\left(\Omega,\bigwedge^\ell\right)\to\mathcal{D}'\left(\Omega,\bigwedge^{\ell+1}\right)$ for $\ell=0,1,\cdots,n$. Its formal adjoint operator $d^*:\mathcal{D}'\left(\Omega,\bigwedge^{\ell+1}\right)\to\mathcal{D}'\left(\Omega,\bigwedge^\ell\right)$ is given by $d^*=(-1)^{n\ell+1}*d*$ on $\mathcal{D}'\left(\Omega,\bigwedge^{\ell+1}\right)$, $\ell=0,1,\cdots,n$. The well-known Poincaré Lemma states that $d\circ d=0$. It is easy to see that $d^*\circ d^*=0$ as well.

A differential ℓ-form $u\in\mathcal{D}'\left(\Omega,\bigwedge^\ell\right)$ is called a closed form if $du=0$ in Ω. It is called exact if there exists a differential form $\alpha\in\mathcal{D}'\left(\Omega,\bigwedge^{\ell-1}\right)$ such that $u=d\alpha$. Poincaré Lemma implies that exact forms are closed. Similarly, a differential ℓ-form $v\in\mathcal{D}'\left(\Omega,\bigwedge^\ell\right)$ is called a coclosed form if $d^*v=0$. It is called coexact if there exists a differential $(\ell+1)$-form $\beta\in\mathcal{D}'\left(\Omega,\bigwedge^{\ell+1}\right)$ such that $v=d^*\beta$. Poincaré Lemma implies that coexact forms are coclosed.

Let $G=\left(G^i_j\right)_{1\le i,j\le n}$ be an $n\times n$ matrix. The ℓ-exterior power of G is a linear operator

$$G^\ell_\#:\bigwedge^\ell\to\bigwedge^\ell$$

defined by

$$G^\ell_\#(\alpha_1\wedge\alpha_2\wedge\cdots\wedge\alpha_\ell)=G\alpha_1\wedge G\alpha_2\wedge\cdots\wedge G\alpha_\ell,$$

where $\alpha_1,\alpha_2,\cdots,\alpha_\ell\in\bigwedge^1$. The linear transform $G^\ell_\#$ can be expressed as an $C^\ell_n\times C^\ell_n$ matrix whose entries are $\ell\times\ell$ minors of G and denoted by $G^\ell_\#=\left(\det G^I_J\right)_{C^\ell_n\times C^\ell_n}$, where $I=(i_1,\cdots,i_\ell)$, $J=(j_1,\cdots,j_\ell)$ are ordered ℓ-tuples and

$$\det G^I_J=\det\begin{bmatrix}G^{i_1}_{j_1}, & \cdots & G^{i_1}_{j_\ell}\\ & \cdots & \\ G^{i_\ell}_{j_1}, & \cdots & G^{i_\ell}_{j_\ell}\end{bmatrix}.$$

Definition 1.1 *A pair of differential ℓ-forms $\mathcal{F}=(\mathcal{C},\mathcal{E})\in L^{p'}\left(\Omega,\bigwedge^\ell\right)\times L^{q'}\left(\Omega,\bigwedge^\ell\right)$, $1\le p',q'<\infty$, is called coclosed-exact, if $d^*\mathcal{C}=0$ and there exists a differential $(\ell-1)$-form $u\in\bigwedge^{\ell-1}$ such that $\mathcal{E}=du$. Moreover, the Jacobian associated to the field $\mathcal{F}$ is defined by $\mathcal{J}(x,\mathcal{F})=\langle\mathcal{C},\mathcal{E}\rangle$.*

In much the same way, we can define the closed-coexact fields of differential forms.

Definition 1.2 *A pair of differential ℓ-forms $\mathcal{F} = (\mathcal{C}, \mathcal{E}) \in L^{p'}\left(\Omega, \bigwedge^\ell\right) \times L^{q'}\left(\Omega, \bigwedge^\ell\right)$, $1 \le p', q' < \infty$, is called closed-coexact, if $d\mathcal{C} = 0$ and there exists a differential $(\ell+1)$-form $u \in \bigwedge^{\ell+1}$ such that $\mathcal{E} = d^* u$. The Jacobian associated to the field $\mathcal{F}$ is defined by $\mathcal{J}(x, \mathcal{F}) = \langle \mathcal{C}, \mathcal{E} \rangle$.*

Balls with radius R are denoted by B_R and $B_{\sigma R}$ is the ball with the same center as B_R and $\text{diam}(B_{\sigma R}) = \sigma \text{diam}(B_R)$. The n-dimensional Lebesgue measure of a set $E \subset \mathrm{R}^n$ is denoted by $|E|$. We can find the following result in [1, 2]: Let $Q \subset \mathrm{R}^n$ be a cube or a ball. To each $y \in Q$ there corresponds a linear operator $K_y : C^\infty\left(Q, \bigwedge^\ell\right) \to C^\infty(Q, \bigwedge^{\ell-1})$ defined by

$$(K_y \omega)(x; \xi_1, \xi_2, \cdots, \xi_{\ell-1}) = \int_0^1 t^{\ell-1} \omega(tx + y - ty; x - y, \xi_1, \cdots, \xi_{\ell-1}) dt$$

and the decomposition

$$\omega = d(K_y) + K_y(d\omega).$$

Another linear operator $T_Q : C^\infty\left(Q, \bigwedge^\ell\right) \to C^\infty\left(Q, \bigwedge^{\ell-1}\right)$ is defined by averaging K_y over all points y in Q

$$T_Q \omega = \int_Q \varphi(y) K_y \omega dy,$$

where $\varphi \in C_0^\infty(Q)$ is normalized by $\int_Q \varphi(y) dy = 1$. We define the ℓ-form $\omega_Q \in \mathcal{D}'\left(Q, \bigwedge^\ell\right)$ by

$$\omega_Q = |Q|^{-1} \int_Q \omega(y) dy, \text{ if } \ell = 0, \text{ and } \omega_Q = d(T_Q \omega), \text{ if } \ell = 1, 2, \cdots, n,$$

for all $\omega \in L^p\left(Q, \bigwedge^\ell\right), 1 \le p < \infty$. It is easy to see that ω_Q is exact.

2 Estimates Below the Natural Exponents

In this section, we derive two estimates below the natural exponents for coclosed-exact and closed-coexact fields of differential forms.

In the following, we denote by $c(*, \cdots, *)$ a constant depending only on the quantities $*, \cdots, *$, whose value may be different from line to line.

We begin with a simple consequence of Hölder's inequality. Let $1 < p', q' < \infty$ be a Hölder conjugate pair, $\frac{1}{p'} + \frac{1}{q'} = 1$. For any pair of differential forms $\mathcal{F} = (\mathcal{C}, \mathcal{E})$ with $\mathcal{C} \in L^{p'}\left(B_R, \bigwedge^\ell\right)$, $\mathcal{E} \in L^{q'}\left(B_R, \bigwedge^\ell\right)$, and any test function $\varphi \in C_0^\infty(B_R)$, we have

$$\left| \int_{B_R} \varphi \mathcal{J}(x, \mathcal{F}) dx \right| = \left| \int_{B_R} \varphi \langle \mathcal{C}, \mathcal{E} \rangle dx \right| \le \|\varphi\|_\infty \|\mathcal{C}\|_{p'} \|\mathcal{E}\|_{q'}.$$

In order to exploit certain cancelations in the above integral we now assume $\mathcal{F} = (\mathcal{C}, \mathcal{E})$ be a coclosed-exact pair of differential forms, that is, $d^*\mathcal{C} = 0$ and there exists a differential $(\ell-1)$-form $u \in \bigwedge^{\ell-1}$ such that $\mathcal{E} = du$. Unless otherwise stated, this assumption will remain valid throughout this article. The following theorems are estimates with integrable exponents below the natural ones. For coclosed-exact fields of differential forms, we have

Theorem 2.1 *Let $1 < p', q' < \infty$ be a Hölder conjugate pair, $\frac{1}{p'} + \frac{1}{q'} = 1$, and $1 < r', s' < \infty$ satisfies $\frac{1}{r'} + \frac{1}{s'} = 1 + \frac{1+\varepsilon}{n}$. Then there exists a constant $c = c(n, p', r')$ such that for each test function $\psi \in C_0^\infty(B_R)$, one has*

$$\int_{B_R} \psi^{1-\varepsilon} \frac{\mathcal{J}(x, \mathcal{F})}{|\mathcal{C}|^\varepsilon |\mathcal{E}|^\varepsilon} dx \leq \begin{array}{l} c\varepsilon \|\mathcal{C}\|_{p'(1-\varepsilon)}^{1-\varepsilon} \|d(\psi(u - u_{B_R}))\|_{q'(1-\varepsilon)}^{1-\varepsilon} \\ +c\|\nabla\psi\|_\infty^{1-\varepsilon} \|\mathcal{C}\|_{r'(1-\varepsilon)}^{1-\varepsilon} \|du\|_{s'(1-\varepsilon)}^{1-\varepsilon}, \end{array} \tag{2.1}$$

wherever $0 \leq 2\varepsilon \leq \min\left\{\frac{p'-1}{p'}, \frac{q'-1}{q'}, \frac{r'-1}{r'}, \frac{s'-1}{s'}\right\}$ and $\mathcal{F} = (\mathcal{C}, \mathcal{E}) \in L^{p'(1-\varepsilon)}\left(B_R, \bigwedge^\ell\right) \times L^{q'(1-\varepsilon)}\left(B_R, \bigwedge^\ell\right) \bigcap L^{r'(1-\varepsilon)}\left(B_R, \bigwedge^\ell\right) \times L^{s'(1-\varepsilon)}\left(B_R, \bigwedge^\ell\right)$ a coclosed-exact field of differential ℓ-forms.

For closed-coexact fields of differential forms, we have

Theorem 2.2 *Let $1 < p', q' < \infty$ be a Hölder conjugate pair, $\frac{1}{p'} + \frac{1}{q'} = 1$, and $1 < r', s' < \infty$ satisfies $\frac{1}{r'} + \frac{1}{s'} = 1 + \frac{1+\varepsilon}{n}$. Then there exists a constant $c = c(n, p', r')$ such that (2.1) holds for each test function $\varphi \in C_0^\infty(B_R)$, wherever $0 \leq 2\varepsilon \leq \min\left\{\frac{p'-1}{p'}, \frac{q'-1}{q'}, \frac{r'-1}{r'}, \frac{s'-1}{s'}\right\}$ and $\mathcal{F} = (\mathcal{C}, \mathcal{E}) \in L^{p'(1-\varepsilon)}\left(B_R, \bigwedge^\ell\right) \times L^{q'(1-\varepsilon)}\left(B_R, \bigwedge^\ell\right) \bigcap L^{r'(1-\varepsilon)}\left(B_R, \bigwedge^\ell\right) \times L^{s'(1-\varepsilon)}\left(B_R, \bigwedge^\ell\right)$ a closed-coexact field of differential ℓ-forms.*

The key tool used in establishing (2.1) is the stability of the Hodge decomposition theorem under nonlinear perturbations of differential forms, first discovered by Iwaniec [3].

Lemma 2.3 *For $\omega \in L^{r(1-\varepsilon)}\left(R^n, \bigwedge^\ell\right)$, $\varepsilon < \frac{1}{2}$, consider the Hodge decomposition*

$$|\omega|^{-\varepsilon}\omega = d\alpha + d^*\beta, \quad \text{with } \alpha \in L_1^r\left(R^n, \bigwedge^{\ell-1}\right) \text{ and } \beta \in L_1^r\left(R^n, \bigwedge^{\ell+1}\right).$$

If ω is closed, then

$$\|d^*\beta\|_r \leq c(n) r |\varepsilon| \|\omega\|_{r(1-\varepsilon)}^{1-\varepsilon}.$$

If ω is coclosed, then

$$\|d\alpha\|_r \leq c(n) r |\varepsilon| \|\omega\|_{r(1-\varepsilon)}^{1-\varepsilon}.$$

In the proof of Theorem 2.1, we will also need the Poincaré and Sobolev-Poincaré inequalities, which can be found in [2], see also [4, 5].

Lemma 2.4 *Suppose that* $\omega \in \mathcal{D}'\left(B, \bigwedge^\ell\right)$ *and* $d\omega \in L^p(B, \bigwedge^{\ell+1})$, $\ell = 0, 1, \cdots, n$. *Then* $\omega - \omega_B$ *is in* $L^p\left(B, \bigwedge^\ell\right)$ *and we have the following uniform estimate*

$$\left(\int_B |\omega - \omega_B|^p dx\right)^{1/p} \le C(p,n)\, diam(B) \left(\int_B |d\omega|^p dx\right)^{1/p}$$

for B *a cube or a ball in* R^n.

Lemma 2.5 *Suppose that* $\omega \in \mathcal{D}'\left(B, \bigwedge^\ell\right)$ *and* $d\omega \in L^p\left(B, \bigwedge^{\ell+1}\right)$, $\ell = 0, 1, \cdots, n$ *and* $1 < p < n$. *Then* $\omega - \omega_B$ *is in* $L^{np/(n-p)}\left(B, \bigwedge^\ell\right)$ *and we have the following uniform estimate*

$$\left(\int_B |\omega - \omega_B|^{np/(n-p)} dx\right)^{(n-p)/np} \le C(p,n) \left(\int_B |d\omega|^p dx\right)^{1/p} \tag{2.2}$$

for B *a cube or a ball in* R^n.

The following lemma comes from [6], which is an elementary inequality for differential ℓ-forms.

Lemma 2.6 *Suppose that* $X, Y \in \bigwedge^\ell$ *be two differential* ℓ*-forms and* $0 \le \varepsilon < 1$. *Then*

$$\left||X|^{-\varepsilon}X - |Y|^{-\varepsilon}Y\right| \le \frac{2^\varepsilon(1+\varepsilon)}{1-\varepsilon}|X-Y|^{1-\varepsilon}.$$

Proof of Theorem 2.1 We define the values of the coefficients of $\mathcal{C}$ and $\mathcal{E}$ to be 0 outside B_R. Let us decompose, according to Lemma 2.3, with $\omega = \mathcal{C} \in L^{p'(1-\varepsilon)}\left(B_R, \bigwedge^\ell\right)$,

$$\begin{cases} |\mathcal{C}|^{-\varepsilon}\mathcal{C} = d\alpha_1 + d^*\beta_1, \quad \alpha_1 \in L_1^{p'}\left(B_R, \bigwedge^{\ell-1}\right), \beta_1 \in L_1^{p'}\left(B_R, \bigwedge^{\ell+1}\right), \\ \|d\alpha_1\|_{p'} \le c(n)p'|\varepsilon|\|\mathcal{C}\|_{p'(1-\varepsilon)}^{1-\varepsilon}, \end{cases} \tag{2.3}$$

and then with $\omega = d(\psi(u - u_{B_R})) \in L^{q'(1-\varepsilon)}\left(B_R, \bigwedge^\ell\right)$,

$$\begin{cases} |d(\psi(u - u_{B_R}))|^{-\varepsilon} d(\psi(u - u_{B_R})) = d\alpha_2 + d^*\beta_2, \\ \qquad \alpha_2 \in L_1^{q'}\left(B_R, \bigwedge^{\ell-1}\right), \beta_2 \in L_1^{q'}\left(B_R, \bigwedge^{\ell+1}\right), \\ \|d^*\beta_2\|_{q'} \le c(n)q'|\varepsilon|\|d(\psi(u - u_{B_R}))\|_{q'(1-\varepsilon)}^{1-\varepsilon}. \end{cases} \tag{2.4}$$

(2.3) and (2.4) imply

$$\|d^*\beta_1\|_{p'} \le c(n)p'\|\mathcal{C}\|_{p'(1-\varepsilon)}^{1-\varepsilon} \tag{2.5}$$

and

$$\|d\alpha_2\|_{q'} \le c(n)q'\|d(\psi(u-u_{B_R}))\|^{1-\varepsilon}_{q'(1-\varepsilon)} \tag{2.6}$$

respectively.

Let us introduce a differential ℓ-form

$$E = |d(\psi(u-u_{B_R}))|^{-\varepsilon}d(\psi(u-u_{B_R})) - |\psi du|^{-\varepsilon}\psi du,$$

then by Lemma 2.5 one has

$$|E| \le \frac{2^\varepsilon(1+\varepsilon)}{1-\varepsilon}|d\psi\wedge(u-u_{B_R})|^{1-\varepsilon}. \tag{2.7}$$

Since coclosed forms are orthogonal to exact forms, then

$$\begin{aligned}
& \int_{B_R}\psi^{1-\varepsilon}\frac{\langle\mathcal{C},\mathcal{E}\rangle}{|\mathcal{C}|^\varepsilon|\mathcal{E}|^\varepsilon}dx \\
= & \int_{B_R}\langle|\mathcal{C}|^{-\varepsilon}\mathcal{C}, |\psi du|^{-\varepsilon}\psi du\rangle dx \\
= & \int_{B_R}\langle|\mathcal{C}|^{-\varepsilon}\mathcal{C}, |d(\psi(u-u_{B_R}))|^{-\varepsilon}d(\psi(u-u_{B_R})) - E\rangle dx \\
= & \int_{B_R}\langle d\alpha_1+d^*\beta_1, d\alpha_2+d^*\beta_2\rangle dx - \int_{B_R}\langle|\mathcal{C}|^{-\varepsilon}\mathcal{C},E\rangle dx \\
= & \int_{B_R}\langle d\alpha_1, d\alpha_2\rangle dx + \int_{B_R}\langle d^*\beta_1, d^*\beta_2\rangle dx - \int_{B_R}\langle|\mathcal{C}|^{-\varepsilon}\mathcal{C},E\rangle dx \\
= & I_1+I_2+I_3.
\end{aligned} \tag{2.8}$$

Our nearest goal is to estimate $|I_1|, |I_2|$ and $|I_3|$ for sufficiently small ε, say $2\varepsilon \le \min\left\{\frac{p'}{p'-1}, \frac{q'}{q'-1}, \frac{r'}{r'-1}, \frac{s'}{s'-1}\right\}$. $|I_1|$ can be estimated by (2.3) and (2.6) as

$$\begin{aligned}
|I_1| &= \left|\int_{B_R}\langle d\alpha_1, d\alpha_2\rangle dx\right| \le \|d\alpha_1\|_{p'}\|d\alpha_2\|_{q'} \\
&\le c(n,p')|\varepsilon|\|\mathcal{C}\|^{1-\varepsilon}_{p'(1-\varepsilon)}\|d(\psi(u-u_{B_R}))\|^{1-\varepsilon}_{q'(1-\varepsilon)}.
\end{aligned} \tag{2.9}$$

$|I_2|$ can be estimated by (2.4) and (2.5) as

$$\begin{aligned}
|I_2| &= \left|\int_{B_R}\langle d^*\beta_1, d^*\beta_2\rangle dx\right| \le \|d^*\beta_1\|_{p'}\|d^*\beta_2\|_{q'} \\
&\le c(n,p')|\varepsilon|\|\mathcal{C}\|^{1-\varepsilon}_{p'(1-\varepsilon)}\|d(\psi(u-u_{B_R}))\|^{1-\varepsilon}_{q'(1-\varepsilon)}.
\end{aligned} \tag{2.10}$$

$|I_3|$ can be estimated by (2.7) and Lemma 2.4 as

$$\begin{aligned}
|I_3| &= \left|\int_{B_R}\langle|\mathcal{C}|^{-\varepsilon}\mathcal{C},E\rangle dx\right| \\
&\le \frac{2^\varepsilon(1+\varepsilon)}{1-\varepsilon}\int_{B_R}|\mathcal{C}|^{1-\varepsilon}|d\psi\wedge(u-u_{B_R})|^{1-\varepsilon}dx \\
&\le c(n)\|\nabla\psi\|^{1-\varepsilon}_\infty\int_{B_R}|\mathcal{C}|^{1-\varepsilon}|u-u_{B_R}|^{1-\varepsilon}dx \\
&\le c(n)\|\nabla\psi\|^{1-\varepsilon}_\infty\left(\int_{B_R}|\mathcal{C}|^{(1-\varepsilon)r'}dx\right)^{1/r'}\left(\int_{B_R}|u-u_{B_R}|^{r'(1-\varepsilon)/(r'-1)}dx\right)^{(r'-1)/r'} \\
&\le c(n,r')\|\nabla\psi\|^{1-\varepsilon}_\infty\|\mathcal{C}\|^{1-\varepsilon}_{r'(1-\varepsilon)}\|du\|^{1-\varepsilon}_{s'(1-\varepsilon)},
\end{aligned} \tag{2.11}$$

where we recall that $\frac{1}{r'} + \frac{1}{s'} = 1 + \frac{1-\varepsilon}{n}$. Combining (2.8)-(2.11) we arrive at (2.1), completing the proof of Theorem 2.1.
Proof of Theorem 2.2 Similar to the proof of Theorem 2.1.

3 An Application to Weakly Quasiregular Mappings

We now give an application of Theorem 2.1 to quasiregular mappings. Let $\Omega \subset \mathrm{R}^n$, $n \geq 2$, and $f = (f^1, f^2, \cdots, f^n) \in W^{1,r}_{loc}(\Omega, \mathrm{R}^n)$, $1 \leq r < \infty$. The differential $Df(x) : \Omega \to GL(n)$ and its determinant $\mathcal{J}_f(x) = \det Df(x)$ are, therefore, defined almost everywhere in Ω. We assume that $\mathcal{J}_f(x)$ is nonnegative.

Definition 3.1 *A mapping $f \in W^{1,r}_{loc}(\Omega, R^n)$ is said to be weakly K-quasiregular, $1 \leq K < \infty$, if*

$$\max_{|\xi|=1} |Df(x)\xi| \leq K \min_{|\xi|=1} |Df(x)\xi|$$

for almost every $x \in \Omega$. It is called K-quasiregular if r is equal to the dimension of the domain, thus $\mathcal{J}_f(x) \in L^1_{loc}(\Omega)$.

The theory of quasiregular mappings is a central topic in modern analysis with important connections to a variety of topics as elliptic partial differential equations, complex dynamics, differential geometry and calculus of variations; see [7, 8] and the references therein. For the recent developments of quasiregular mapping theory, see [7-12].

If we introduce, for every K-quasiregular mapping f, a metric tensor $G(x)$ on Ω,

$$G(x) = \begin{cases} \mathcal{J}_f^{-2/n}(x) D^t f(x) Df(x), & \text{for } \mathcal{J}_f(x) \neq 0, \\ \mathrm{Id}, & \text{for } \mathcal{J}_f(x) = 0, \end{cases}$$

where $D^t f(x)$ and Id are the transpose of $Df(x)$ and the identity matrix, respectively, then quasiregular mappings are simply weak solutions to the differential system

$$D^t f(x) Df(x) = \mathcal{J}_f^{2/n}(x) G(x),$$

commonly called the n-dimensional Beltrami equation.

Fix an ordered ℓ-tuple $I = (i_1, i_2, \cdots, i_\ell)$ and its complementary $(n-1)$-tuple $J = (j_1, j_2, \cdots, j_{n-\ell})$ ordered in such a way that $dx_I = *dx_J$. Suppose that $r \geq \max\{\ell, n-\ell\}$. To each such pair (I, J) we assign the differential form

$$u_I = f^{i_\ell} df^{i_1} \wedge \cdots \wedge df^{i_{\ell-1}} \in L^{n/(n-1)}_{loc}\left(\Omega, \bigwedge^{\ell-1}\right)$$

and the conjugate form

$$v_J = *f^{j_1} df^{j_2} \wedge \cdots \wedge df^{j_{n-\ell}} \in L^{n/(n-1)}_{loc}\left(\Omega, \bigwedge^{\ell+1}\right).$$

The degree of local integrability is verified by the Sobolev embedding theorem. Clearly,

$$du_I = (-1)^{\ell-1} df^{i_1} \wedge \cdots \wedge df^{i_\ell} \in L^1_{loc}\left(\Omega, \bigwedge^{\ell}\right)$$

and

$$d^* v_J = (-1)^{\ell+1} * df^{j_1} \wedge \cdots \wedge df^{j_{n-\ell}} \in L^1_{loc}\left(\Omega, \bigwedge^{\ell}\right).$$

From [3], we know that the differential forms $du_I, d^*v_j \in L^1_{loc}(\Omega, \bigwedge^\ell)$ satisfy the p-harmonic and the conjugate q-harmonic equations

$$d^*\mathcal{A}(x, du_I) = 0 \tag{3.1}$$

$$d\mathcal{A}^{-1}(x, d^*v_J) = 0 \tag{3.2}$$

respectively, where

$$\mathcal{A}(x,\xi) = \langle (G^\ell_\#)^{-1}(x)\xi, \xi\rangle^{(p-2)/2}(G^\ell_\#)^{-1}(x)\xi, \quad p = \frac{n}{\ell},$$

$$\mathcal{A}^{-1}(x,\xi) = \langle (G^\ell_\#)(x)\xi, \xi\rangle^{(q-2)/2}(G^\ell_\#)(x)\xi, \quad q = \frac{n}{n-\ell},$$

and the following estimates hold

$$\langle \mathcal{A}(x, du_I), du_I\rangle \geq c_1 |du_I|^p, \tag{3.4}$$

$$|\mathcal{A}(x, du_I)| \leq c_2 |du_I|^{p-1}. \tag{3.5}$$

We recall a famous regularity result due to T.Iwaniec, see [3, Theorem 3].

Theorem 3.2 *There exist exponents $q = q(n,K) < n < p(n,K) = p$ such that every weakly K-quasiregular mapping of class $W^{1,q}_{loc}(\Omega, R^n)$ belongs to $W^{1,p}_{loc}(\Omega, R^n)$ and so is K-quasiregular.*

We now give an alternative proof of Theorem 3.1 by using Theorem 2.1. Similarly, Theorem 3.1 can also be proved by using Theorem 2.2.

An examination of [3] reveals that Theorem 3.1 is based on a weak reverse Hölder inequality. Instead of rewriting all the needed steps, we only prove the following lemma, which is sufficient to the proof of Theorem 3.1.

Lemma 3.3 *For every weakly K-quasiregular mapping of class $W^{1,n(1-\varepsilon)}_{loc}(\Omega, R^n)$, we have the weakly reverse Hölder inequality*

$$\fint_{B_{R/2}} |du_I|^{p(1-\varepsilon)}dx \le \theta \fint_{B_R} |du_I|^{p(1-\varepsilon)}dx + \left(\fint_{B_R} |du_I|^{\frac{np(1-\varepsilon)}{n+1-\varepsilon}}dx\right)^{\frac{n+1-\varepsilon}{n}}. \quad (3.6)$$

provided that ε small enough, where $\fint_{B_R} = \frac{1}{|B_R|}\int_B$ is the integral mean over B_R.

Proof. For quasiregular mapping $f \in W^{1,n(1-\varepsilon)}_{loc}(\Omega, \mathrm{R}^n)$, we introduce two differential ℓ-forms $\mathcal{C} = \mathcal{A}(x, du_I)$ and $\mathcal{E} = du_I$, then by (3.1), it is obvious that $\mathcal{F} = (\mathcal{C}, \mathcal{E})$ is a coclosed-exact pair. For $B_R \subset\subset \Omega$, take $\psi \in C^\infty_0(B_R)$ such that $0 \le \psi \le 1$, $\psi \equiv 1$ on $B_{R/2}$ and $|\nabla\psi| \le \frac{c(n)}{R}$. Then by (3.4) and (3.5),

$$\int_{B_R} \psi^{1-\varepsilon}\frac{\mathcal{J}(x,\mathcal{F})}{|\mathcal{C}|^\varepsilon|\mathcal{E}|^\varepsilon}dx = \int_{B_R} \psi^{1-\varepsilon}\frac{\langle\mathcal{A}(x,du_I), du_I\rangle}{|\mathcal{A}(x,du_I)|^\varepsilon|du_I|^\varepsilon}dx \ge c\int_{B_{R/2}} |du_I|^{p(1-\varepsilon)}dx. \quad (3.7)$$

Take $p' = \frac{p}{p-1}$ and $q' = p$ we obtain from Lemma 2.4 that

$$\begin{aligned} & \varepsilon\|\mathcal{C}\|^{1-\varepsilon}_{p'(1-\varepsilon)}\|d(\psi(u_I - (u_I)_{B_R}))\|^{1-\varepsilon}_{q'(1-\varepsilon)} \\ \le\ & \varepsilon\|\mathcal{C}\|^{1-\varepsilon}_{p'(1-\varepsilon)}\left[\|\psi du_I\|^{1-\varepsilon}_{q'(1-\varepsilon)} + \|d\psi \wedge (u_I - (u_I)_{B_R})\|^{1-\varepsilon}_{q'(1-\varepsilon)}\right] \\ \le\ & c\varepsilon\|\mathcal{C}\|^{1-\varepsilon}_{p'(1-\varepsilon)}\left[\|du_I\|^{1-\varepsilon}_{q'(1-\varepsilon)} + \frac{1}{R^{1-\varepsilon}}\|(u_I - (u_I)_{B_R})\|^{1-\varepsilon}_{q'(1-\varepsilon)}\right] \\ \le\ & c\varepsilon\|du_I\|^{(p-1)(1-\varepsilon)}_{p(1-\varepsilon)}\left[\|du_I\|^{1-\varepsilon}_{p(1-\varepsilon)} + \frac{1}{R^{1-\varepsilon}}\|(u_I - (u_I)_{B_R})\|^{1-\varepsilon}_{p(1-\varepsilon)}\right] \\ \le\ & c\varepsilon\|du_I\|^{p(1-\varepsilon)}_{p(1-\varepsilon)}. \end{aligned} \quad (3.8)$$

Take $r' = \frac{np}{(p-1)(n+1-\varepsilon)}$ and $s' = \frac{np}{n+1-\varepsilon}$, we obtain

$$\begin{aligned} & \|\nabla\psi\|^{1-\varepsilon}_\infty\|\mathcal{C}\|^{1-\varepsilon}_{r'(1-\varepsilon)}\|du_I\|^{1-\varepsilon}_{s'(1-\varepsilon)} \\ \le\ & \frac{c}{R^{1-\varepsilon}}\|du_I\|^{(p-1)(1-\varepsilon)}_{\frac{np(1-\varepsilon)}{n+1-\varepsilon}}\|du_I\|^{1-\varepsilon}_{\frac{np(1-\varepsilon)}{n+1-\varepsilon}} \\ =\ & \frac{c}{R^{1-\varepsilon}}\|du_I\|^{p(1-\varepsilon)}_{\frac{np(1-\varepsilon)}{n+1-\varepsilon}}. \end{aligned} \quad (3.9)$$

Combining (2.1) with (3.7), (3.8) and (3.9) we get that

$$\int_{B_{R/2}} |du_I|^{p(1-\varepsilon)}dx \le c\varepsilon\int_{B_R} |du_I|^{p(1-\varepsilon)}dx + \frac{c}{R^{1-\varepsilon}}\left(\int_{B_R} |du_I|^{\frac{np(1-\varepsilon)}{n+1-\varepsilon}}dx\right)^{\frac{n+1-\varepsilon}{n}}.$$

Divide both sides of the above inequality by $|B_{R/2}| = \omega_n(R/2)^n$ we obtain

$$\fint_{B_{R/2}} |du_I|^{p(1-\varepsilon)}dx \le c\varepsilon\fint_{B_R} |du_I|^{p(1-\varepsilon)}dx + c\left(\fint_{B_R} |du_I|^{\frac{np(1-\varepsilon)}{n+1-\varepsilon}}dx\right)^{\frac{n+1-\varepsilon}{n}}.$$

Take ε small enough such that $\theta = c\varepsilon < 1$, we arrive at (3.7). Lemma 3.3 has been proved.

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