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On the Oscillation of Nonlinear Fractional Difference Equations

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Abstract

In this paper, we study oscillatory behavior of the fractional difference equations of the following form

$$\Delta\left(p(t)(\Delta^{\alpha}x(t))^{\gamma}\right)+q(t)f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)=0, t\in N_{t_0+1-\alpha},$$

where Δ^{α} denotes the Riemann-Liouville difference operator of order α, $0<\alpha\leq 1$ and $\gamma>0$ is a quotient of odd positive integers. We establish some oscillation criteria for the above equation by using Riccati transformation technique and some Hardy type inequalities. An example is provided to illustrate our main results.

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1 Introduction

Oscillatory behavior of fractional differential equations have been dealt by several authors, see [2]-[9] and the theory of fractional differential equations is presented in books, see [16]-[18]. But the qualitative properties of fractional difference equations are studied by few authors, see [10]-[15]. Motivated by [3] and [9], we study the oscillatory behavior of the following fractional difference equation of the form

$$\Delta\left(p(t)(\Delta^{\alpha}x(t))^{\gamma}\right)+q(t)f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)=0, t\in N_{t_0+1-\alpha}, \quad (1)$$

for $0 < \alpha \leq 1$. Here Δ^{α} denotes the Riemann-Liouville difference operator, $\gamma > 0$ is a quotient of odd positive integers. In this paper we assume the following conditions.
(H_1). $p(t)$ and $q(t)$ are positive sequences and $f : R \to R$ is a continuous function such that$f(x)/(x^n) \geq k$ for a certain constant $k > 0$ and for all $x \neq 0$.

A solution $x(t)$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

2 Preliminaries and Basic Lemmas

In this section, we present some preliminary results of discrete fractional calculus, which will be used throughout this paper.

Definition 2.1 *(see [12]) Let $\nu > 0$. The ν-th fractional sum f is defined by*

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s),$$

where f is defined for $s \equiv a \mod (1)$ and $\Delta^{-\nu} f$ is defined for $t \equiv (a+\nu) \mod (1)$ and $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$. The fractional sum $\Delta^{-\nu} f$ maps functions defined on N_a to functions defined on N_{a+v}.

Definition 2.2 *(see [12]) Let $\mu > 0$ and $m - 1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$. Set $\nu = m - \mu$. The μ-th order Riemann-Liouville fractional difference is defined as*

$$\Delta^{\mu} f(t) = \Delta^{m-\nu} f(t) = \Delta^{m} \Delta^{-\nu} f(t).$$

Lemma 2.3 *Let $x(t)$ be a solution of (1) and let*

$$G(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s) \tag{2}$$

then

$$\Delta(G(t)) = \Gamma(1-\alpha) \Delta^{\alpha}(x(t)). \tag{3}$$

Proof:

$$G(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s) = \sum_{s=t_0}^{t-(1-\alpha)} (t-s-1)^{(1-\alpha)-1} x(s)$$
$$= \Gamma(1-\alpha) \Delta^{-(1-\alpha)} x(t),$$

which implies

$$\Delta(G(t)) = \Gamma(1-\alpha)\Delta\Delta^{-(1-\alpha)}x(t) = \Gamma(1-\alpha)\Delta^{\alpha}x(t).$$

In order to discuss our results in Section 3, now we state the following lemma.

Lemma 2.4 *(Hardy et al. see [1]) If X and Y are nonnegative, then*

$$mXY^{m-1} - X^m \le (m-1)Y^m \quad \textit{for} \quad m > 1 \tag{4}$$

where equality holds if and only if $X = Y$.

3 Main Results

Theorem 3.1 *Suppose that (H_1) and*

$$\sum_{s=t_0}^{\infty} p^{-1/\gamma}(s) = \infty \tag{5}$$

holds. Furthermore, assume that there exists a positive sequence $b(t)$ such that

$$\limsup_{t\to\infty} \sum_{s=t_0}^{t-1} \left(kb(s)q(s) - \frac{(\Delta b_+(s))^2}{4b^2(s+1)R(s)} \right) = \infty, \tag{6}$$

where $R(t) = \frac{b(t)\Gamma(1-\alpha)^{\gamma}}{b^2(t+1)p(t)}$ and $\Delta b_+(s) = \max[\Delta b(s), 0]$. Then every solution of (1) is oscillatory.

Proof: Suppose the contrary that $x(t)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1). Then there exists $t_1 > t_0$ such that

$$x(t) > 0 \quad \text{and} \quad G(t) > 0 \quad \text{for} \quad t \ge t_1, \tag{7}$$

where G is defined as in (2). Therefore, it follows from (1) that

$$\Delta\left(p(t)(\Delta^{\alpha}x(t))^{\gamma}\right) = -q(t)f\left(G(t)\right) < 0 \quad \text{for} \quad t \ge t_1. \tag{8}$$

Thus $p(t)(\Delta^{\alpha}x(t))^{\gamma}$ is an eventually non increasing sequence. First we show that $p(t)(\Delta^{\alpha}x(t))^{\gamma}$ is eventually positive. Suppose there is an integer $t_1 > t_0$ such that $p(t_1)(\Delta^{\alpha}x(t_1))^{\gamma} = c < 0$ for $t \ge t_1$, so that

$$p(t)(\Delta^{\alpha}x(t))^{\gamma} \le p(t_1)(\Delta^{\alpha}x(t_1))^{\gamma} = c < 0$$

which implies that

$$\frac{\Delta G(t)}{\Gamma(1-\alpha)} = \Delta^\alpha x(t) \le c^{1/\gamma} p^{-1/\gamma}(t) \quad \text{for} \quad t \ge t_1.$$

Summing both sides of the last inequality from t_1 to $t-1$, we get

$$G(t) \le G(t_1) + \Gamma(1-\alpha)c^{1/\gamma} \sum_{s=t_1}^{t-1} p^{-1/\gamma}(s) \to -\infty \quad \text{as} \quad t \to \infty, \tag{9}$$

which contradicts the fact that $G(t) > 0$. Hence $p(t)(\Delta^\alpha x(t))^\gamma > 0$ is eventually positive. Define the function $w(t)$ by the Riccati substitution

$$w(t) = b(t)\frac{p(t)(\Delta^\alpha x(t))^\gamma}{G^\gamma(t)} \quad \text{for} \quad t \ge t_1. \tag{10}$$

Then we have $w(t) > 0$ for $t \ge t_1$. It follows that

$$\begin{aligned}
\Delta w(t) =& \Delta b(t)\frac{w(t+1)}{b(t+1)} \\
&+ \frac{b(t)\Delta(p(t)(\Delta^\alpha x(t))^\gamma)G^\gamma(t+1) - b(t)p(t+1)(\Delta^\alpha x(t+1))^\gamma \Delta G^\gamma(t)}{G^\gamma(t+1)G^\gamma(t)} \\
\le& \Delta b_+(t)\frac{w(t+1)}{b(t+1)} - \frac{b(t)q(t)f(G(t))}{G^\gamma(t)} - \frac{b(t)p(t+1)(\Delta^\alpha x(t+1))^\gamma \Delta G^\gamma(t)}{G^{2\gamma}(t+1)}.
\end{aligned}$$

Now using the inequality (see [1])

$$x^\beta - y^\beta \ge (x-y)^\beta,$$

we have

$$\begin{aligned}
p(t)(\Delta^\alpha x(t))^\gamma \ge& p(t+1)(\Delta^\alpha x(t+1))^\gamma \\
(\Delta^\alpha x(t))^\gamma \ge& \frac{p(t+1)}{p(t)}(\Delta^\alpha x(t+1))^\gamma.
\end{aligned}$$

Using the above inequality

$$\begin{aligned}
\Delta w(t) \le& \Delta b_+(t)\frac{w(t+1)}{b(t+1)} - kb(t)q(t) - \frac{b(t)p(t+1)(\Delta^\alpha x(t+1))^\gamma (\Delta G(t))^\gamma}{G^{2\gamma}(t+1)} \\
\le& \Delta b_+(t)\frac{w(t+1)}{b(t+1)} - kb(t)q(t) - \frac{b(t)p(t+1)(\Delta^\alpha x(t+1))^\gamma (\Gamma(1-\alpha)\Delta^\alpha x(t))^\gamma}{\frac{b^2(t+1)p^2(t+1)(\Delta^\alpha x(t+1))^{2\gamma}}{w(t+1)^2}} \\
\le& \Delta b_+(t)\frac{w(t+1)}{b(t+1)} - kb(t)q(t) - \frac{b(t)\Gamma(1-\alpha)^\gamma \frac{p(t+1)}{p(t)}(\Delta^\alpha x(t+1))^\gamma}{\frac{b^2(t+1)p(t+1)(\Delta^\alpha x(t+1))^\gamma}{w(t+1)^2}} \\
\le& \Delta b_+(t)\frac{w(t+1)}{b(t+1)} - kb(t)q(t) - \frac{b(t)\Gamma(1-\alpha)^\gamma}{b^2(t+1)p(t)}w(t+1)^2 \\
=& \Delta b_+(t)\frac{w(t+1)}{b(t+1)} - kb(t)q(t) - R(t)w(t+1)^2
\end{aligned} \tag{11}$$

where $R(t) = \frac{b(t)\Gamma(1-\alpha)^{\gamma}}{b^2(t+1)p(t)}$. We now set

$$X = \sqrt{R(t)}w(t+1) \quad \text{and} \quad Y = \frac{\Delta b_+(t)}{2b(t+1)\sqrt{R(t)}}.$$

Using Lemma 2.4 and put $m = 2$, we obtain

$$\begin{aligned} 2\left(\sqrt{R(t)}w(t+1)\right)\left(\frac{\Delta b_+(t)}{2b(t+1)\sqrt{R(t)}}\right)^{(2-1)} &- \left(\sqrt{R(t)}w(t+1)\right)^2 \\ &\leq (2-1)\left(\frac{\Delta b_+(t)}{2b(t+1)\sqrt{R(t)}}\right)^2 \\ &= \frac{(\Delta b_+(t))^2}{4b^2(t+1)R(t)}. \end{aligned}$$

From (11), we conclude that

$$\Delta w(t) \leq -kb(t)q(t) + \frac{(\Delta b_+(t))^2}{4b^2(t+1)R(t)}.$$

Summing the above inequality from t_1 to $t-1$, we have

$$\sum_{s=t_1}^{t-1}\left(kb(s)q(s) - \frac{(\Delta b_+(s))^2}{4b^2(s+1)R(s)}\right) \leq w(t_1) - w(t) \leq w(t_1) < \infty, \quad \text{for} \quad t \geq t_1.$$

Letting $t \to \infty$, we get

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1}\left(kb(s)q(s) - \frac{(\Delta b_+(s))^2}{4b^2(s+1)R(s)}\right) \leq w(t_1) < \infty,$$

which contradicts (6). The proof is complete.

Theorem 3.2 *Suppose that* (H_1) *and* $\sum_{s=t_0}^{\infty} p^{-1/\gamma}(s) = \infty$ *hold. Furthermore, assume that there exists a positive sequence* $b(t)$ *such that*

$$\begin{aligned} H(t,t) = 0 \quad &for \quad t \geq t_0 \quad H(t,s) > 0 \quad t > s \geq t_0 \\ \Delta_2 H(t,s) = H(t,s+1) - H(t,s) \leq 0 \quad &for \quad t > s \geq t_0. \end{aligned}$$

If

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t-1}\left(b(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4kH(t,s)R(s)}\right) = \infty, \tag{12}$$

where $h_+(t,s) = \Delta_2 H(t,s) + \frac{H(t,s)\Delta b_+(s)}{b(s+1)}$ *and* $\Delta b_+(s) = \max[\Delta b(s), 0]$. *Then every solution of (1) is oscillatory.*

Proof: Suppose the contrary that $x(t)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1). We proceed as in the proof of Theorem (3.1) to get (11) hold. Multiplying (11) by $H(t,s)$ and summing from t_1 to $t-1$, we obtain

$$\begin{aligned}\sum_{s=t_1}^{t-1} kb(s)q(s)H(t,s) \leq & -\sum_{s=t_1}^{t-1} H(t,s)\Delta w(s) + \sum_{s=t_1}^{t-1} H(t,s)\Delta b_+(s)\frac{w(s+1)}{b(s+1)} \\ & -\sum_{s=t_1}^{t-1} H(t,s)R(s)w^2(s+1)\end{aligned} \tag{13}$$

Using the summation by parts formula, we obtain

$$\begin{aligned}-\sum_{s=t_1}^{t-1} H(t,s)\Delta w(s) =& -\left[H(t,s)w(s)\right]_{s=t_1}^{t} + \sum_{s=t_1}^{t-1} w(s+1)\Delta_2 H(t,s) \\ =& H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} w(s+1)\Delta_2 H(t,s).\end{aligned} \tag{14}$$

Now, we have

$$\begin{aligned}k\sum_{s=t_1}^{t-1} b(s)q(s)H(t,s) \leq & H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} w(s+1)\Delta_2 H(t,s) \\ & + \sum_{s=t_1}^{t-1} H(t,s)\Delta b_+(s)\frac{w(s+1)}{b(s+1)} - \sum_{s=t_1}^{t-1} H(t,s)R(s)w^2(s+1) \\ \leq & H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1}\left(\Delta_2 H(t,s) + \frac{H(t,s)\Delta b_+(s)}{b(s+1)}\right)w(s+1) \\ & -\sum_{s=t_1}^{t-1} H(t,s)R(s)w^2(s+1) \\ \leq & H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1}\left(h_+(t,s)w(s+1) - H(t,s)R(s)w^2(s+1)\right)\end{aligned} \tag{15}$$

where $h_+(t,s) = \Delta_2 H(t,s) + \frac{H(t,s)\Delta b_+(s)}{b(s+1)}$ is defined as in Theorem 3.2. Set

$$X = \sqrt{H(t,s)R(s)}\,w(s+1) \quad \text{and} \quad Y = \frac{h_+(t,s)}{2\sqrt{H(t,s)R(s)}}$$

Using the Lemma 2.4 with $m = 2$, we get

$$
\begin{aligned}
2\left(\sqrt{H(t,s)R(s)}w(s+1)\right)\left(\frac{h_+(t,s)}{2\sqrt{H(t,s)R(s)}}\right)^{(2-1)} &- \left(\sqrt{H(t,s)R(s)}w(s+1)\right)^2 \\
&\leq (2-1)\left(\frac{h_+(t,s)}{2\sqrt{H(t,s)R(s)}}\right)^2 \\
&= \frac{h_+^2(t,s)}{4H(t,s)R(s)}.
\end{aligned}
$$

From equation (15), we have $\Delta_2 H(t,s) \leq 0$ for $t > s \geq t_0$, $0 < H(t,t_1) \leq H(t,t_0)$ for $t > t_1 \geq t_0$

$$
\sum_{s=t_1}^{t-1} b(s)q(s)H(t,s) \leq k^{-1}H(t,t_1)w(t_1) + k^{-1}\sum_{s=t_1}^{t-1}\frac{h_+^2(t,s)}{4kH(t,s)R(s)}
$$

$$
\begin{aligned}
\sum_{s=t_1}^{t-1}\left(b(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4kH(t,s)R(s)}\right) &\leq k^{-1}H(t,t_1)w(t_1) \\
&\leq k^{-1}H(t,t_0)w(t_1).
\end{aligned}
$$

Since $0 < H(t,s) \leq H(t,t_0)$ for $t > s \geq t_0$, we have $0 < \frac{H(t,s)}{H(t,t_0)} \leq 1$ for $t > s \geq t_0$. Hence it follows from that

$$
\begin{aligned}
&\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left(b(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4H(t,s)R(s)}\right) \\
&= \frac{1}{H(t,t_0)}\sum_{s=t_0}^{t_1-1}\left(b(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4H(t,s)R(s)}\right) \\
&+ \frac{1}{H(t,t_0)}\sum_{s=t_1}^{t-1}\left(b(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4H(t,s)R(s)}\right) \\
&\leq \frac{1}{H(t,t_0)}\sum_{s=t_0}^{t_1-1} b(s)q(s)H(t,s) + k^{-1}w(t_1) \\
&\leq \sum_{s=t_0}^{t_1-1} b(s)q(s) + k^{-1}w(t_1).
\end{aligned}
$$

Letting $t \to \infty$, we have

$$
\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left(b(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4kH(t,s)R(s)}\right) \leq \sum_{s=t_0}^{t_1-1} b(s)q(s) + k^{-1}w(t_1) < \infty,
$$

which is a contradiction to (12). The proof is complete.

Example 3.3 *Consider the fractional differential equation*

$$\Delta\left(t^{\gamma-1}(\Delta^{\alpha}(x(t)))^{\gamma}\right)+\frac{1}{t^{2}}\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)^{\gamma}=0, \tag{16}$$

where $\alpha = 0.5$, $\gamma > 0$ *is a quotient of odd positive integers and* $k = 1$, $p(t) = t^{\gamma-1}$, $q(t) = \frac{1}{t^2}$. *Since*

$$\sum_{s=t_0}^{\infty} p^{(-1/\gamma)}(s) = \sum_{s=t_0}^{\infty} \frac{1}{t^{1-1/\gamma}} = \infty,$$

we find that (H_1) *and (5) hold. We will apply Theorem (3.1) and it remains to satisfy condition (6). Taking* $b(s) = s$, *we obtain*

$$\limsup_{t\to\infty}\sum_{s=t_0}^{t-1}\left(kb(s)q(s)-\frac{(\Delta b_{+}(s))^{2}}{4b^{2}(s+1)R(s)}\right)=\limsup_{t\to\infty}\sum_{s=t_0}^{t-1}\frac{1}{s}\left(1-\frac{s^{\gamma-1}}{4\left(\sqrt{\pi}\right)^{\gamma}}\right)=\infty$$

which implies that (6) holds. Therefore, by Theorem (3.1) every solution of (16) is oscillatory.

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