Mathematica Aeterna, Vol. 4, 2014, no. 2, 101 - 117

A non-newtonian examination of the theory of exogenous economic growth

Diana Andrada Filip

Department of Statistics, Forcasting and Mathematics
Babeş-Bolyai University of Cluj-Napoca, Romania
LEO/CNRS — UMR 7322 — Orléans, France
diana.filip@econ.ubbcluj.ro

Cyrille Piatecki

LEO/CNRS — UMR 7322 — Orléans, France
ALPTIS
cyrille.piatecki@univ-orleans.fr

Abstract

The development of the newtonian calculus has unexpectedly be a look-in in such a way that the additive derivative approach of Newton and Leibnitz could have been replaced by a multiplicative one, more adapted to growth phenomenon. In this paper, we have tried to present how a non-newtonian calculus could be applied to re-postulate and analyse the neoclassical exogenous growth model.

Mathematics Subject Classification: 91B02, 91B62

Keywords: non-newtonian calculus, theory of economic growth

1 Introduction

The development of the occidental science has been an incredible success through the introduction of the newtonian calculus. One could say that until the XXth century when discontinuous methods became available, without the derivative nothing of great could have been accomplished.

In the social science, for a very long time, the derivative has been the center of nearly all the analysis. But in an unexpected way, it appears that we can develop another approach which seems to threat more realistically growth phenomenon which are involved in the model of economic growth.

The change of paradigm is to consider that the variations are more naturally taken into account if the deviations are measured by ratios instead of differences. But even if Galileo has discussed briefly such an opportunity, it is not until 1972 that Grossman and Katz [13] imagine a non-newtonian calculus.

In the first section of this paper, we will show how the newtonian calculus is a look-in in the same way as industrial economists described for instance the adoption of the Qwerty keyboard — see David [8] — and we will introduce a non-newtonian calculus of multiplicative type. In the second section, we will justify why the product calculus is the good way to be taken into account for analyse growth phenomenon — see section 3. We will present some properties and derivative rules for this calculus.

In the third section, we will review the Solow-Swan exogenous growth model which will be modified in the fourth section in a non-newtonian way. The contributions of the authors are the reformulation of the ⋆*differential equation* of accumulation of the capital by considering that the increase in the capital equalizes the ratio between investment and obsolescence and the definition of the growth of the labor force. In the curse of this new approach, we will present a non-newtonian Euler's theorem for homogeneous function using the ⋆*partial derivative* for which we will give a new formula. To finish, we will compare the two ODE which are obtained in the two paradigms.

2 An unexpected locked-in

First of all be sure that the subject of this article is a reexamination of the theory of exogenous growth as exposed by Solow [18] and Swan [20]. But, the point of view we will adopt is so unexpected that it need a profound explanation before to begin to describe and analyze the model. This is because every dynamical continuous macro-economic model rest on calculus and that unexpectedly calculus is a lock-in in the very sens of David [8] and Arthur [1], that is to say that the past events and choices have influenced the way we think and make use of the calculus.

If with Cajori [4], one must recognize that calculus is not the work of a lone individual but the collective achievement of numerous author like Cavalieri, Roberval, Fermat, Descartes, Wallis, and others who each contributed to the new geometry, it stays on the stage that the main achievement in this subject has been accomplish nearly simultaneously by Isaac Newton in its *theory of fluxions* and by Gottfried Wilhelm von Leibnitz through its differential quotient and integral which are a one and unique concept. As Grabiner [11] remarks "*they devised a notation for these concepts which made the calculus an algorithm: the methods not only worked, but were easy to use. Their notations had* *great heuristic power, and we still use Leibniz's dy/dx and $\int ydx$ and Newton's x today. Third, both men realized that the basic processes of finding tangents and areas, that is, differentiating and integrating, are mutually inverse what we now call the Fundamental Theorem of Calculus."*

Because for the layman and even for the applied scientist, calculus is so hard to acquire after that Cauchy construct the rigourous $\epsilon - \delta$ definition of continuity and derivative, it's very difficult to understand that the definition of the derivative of a fonction $f : \mathbb{R} \rightarrow \mathbb{R}$ which is :

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = f'(t)$$

is a lock-in. In this case, there is a profound type of inertia, because for at least two century nobody developed any questioning about the way things could have done showing that even science could be characterized by a sociological path dependency exactly in the same way as institution are selected as shown by North [16], technology chosen — David [8] and Arthur [1] — the pest where controlled — Cowan and Gunby [7] — or language and law formed — Hataway [14].

In fact nothing could have prevent the use of

$$\lim_{h \to 0} \left(\frac{f(t+h)}{f(t)} \right)^{\frac{1}{h}} = f^{\star}(t)$$

as a natural definition of the derivative — at least for positive functions. Before Volterra [22], whose definition is not this one, no mention has been made of an alternative to the newtonian calculus. In fact, one must acknowledge that it's only under the effort of Grossman and Katz [13] — look also Grossman [12] — that such a non-newtonian calculus emerged to give a natural answer to many growth phenomenon. But even if it was regarded as a major advance by many scientist like Kenneth Arrow, this major little book stays as a curiosity in the shelves of some university libraries[1].

But, in the end of the last century, some presentations of what one can call the *product calculus* or *multiplicative* become available if not perfectly publicised. For instance, we can now find Stanley [19], Campbell [5] and the incredibly powerful Bashirov et al. [2] which gives nearly all what is necessary to understand and to know about multiplicative calculus[2] from

[1]Long ago, one of the authors of this paper, found it in the Orléans University economic library without even knowing which colleague has had the curiosity to ask for the book.

[2]Which is only one of the many other calculus which according to Grossman and Katz [13] could have been developed as the *anageometric calculus*, the *bigeometric calculus*, the *quadratic family of calculi*, the *harmonic family of calculi*. . . . Note that for Grossman and Katz [13], what we call now multiplicative calculus was better named *geometric calculus*.

the basic properties to the extension of the Euler-Lagrange equation of the calculus of variation.

Up to there one can asked why to introduce a non-newtonian calculus. In fact, if one realised that without the standard newtonian calculus there will not have been the so incredible run up of all the sciences which put us on Newton and Leibnitz giant shoulders, we will loose the point that the newtonian relatively fail when we want to describe growth phenomenon which are essentially of a multiplicative nature and not, as they are mainly dealt as additive.

3 Some justification and properties of the product calculus

Since Malthus [15], the standard way to introduce growth is to go through the resolution of the linear differential equation :

$$\frac{\dot{x}(t)}{x(t)} = n$$

which is universally interpreted as telling that the growth rate of the state variable $x(t)$ is a constant[3] equal to n. But this is an universal abuse of language because n is not a rate of growth. It's a *growth constant*. The rate of growth is by nature a multiplicative phenomenon. In fact, we must start with a linear recurrence equation of the type :

$$x(t+1) = \alpha x(t) \iff \frac{x(t+1)}{x(t)} = \alpha$$

because it describe the ratio between the final and the initial values of the state variable and ask what happens if the interval of change of $x(t)$ goes far beyond the unity. The newtonian procedure is to divide the unity in $1/h$ subintervals of length h in such a way to conserve the unit length and to attribute to each interval a share of α proportional to it — *i.e.* : we substitue αh to α — see the Figure 1.

But this procedure does not work in the intuitive structure. In fact, it is mandatory to redefine α as $1 + \alpha'$ in such a way to substitute $1 + \alpha' h$ which gives :

$$x(t+h) = (1+\alpha' h)x(t) \iff \frac{x(t+h) - x(t)}{h} = \alpha' x(t)$$

[3] Even if this approach may be controversial — look for instance to Verhulst [21] — when we look at population growth, it's apparently not competed when we deal with compound interest.

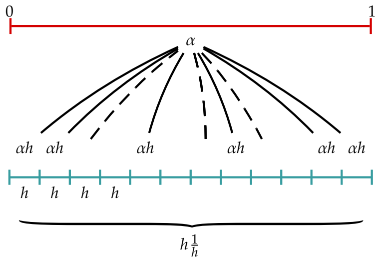

Figure 1: NEWTONIAN ALLOCATION OF α ON SUB-INTERVALS

After the passage to limits, we find that $\dot{x}(t) = \alpha' x(t)$ which is not expressed in term of α but in terms of α'. So, it appears that the newtonian procedure is not the correct one even if there is nothing particularly wrong about it. The good procedure is to affect to each subinterval not αh but α^h because, in that case, we find that :

$$x(t+h) = \alpha^h x(t) \iff \left(\frac{x(t+h)}{x(t)}\right)^{\frac{1}{h}} = \alpha$$

If the limit of the expression exists when $h \to 0$, we will note it by $x^\star(t) = \alpha$. In such a way, one have defined a continuous time multiplicative factor of growth which does not encompass the necessity to go from α to α'.

We can see that

$$x(t+(h_1+h_2)) = x((t+h_1)+h_2) = \alpha^{h_2} x(t+h_1) = \alpha^{h_2}\left(\alpha^{h_1}x(t)\right) = \alpha^{(h_1+h_2)}x(t)$$

which demonstrate the coherence of this approach. So our growth function is continuous, positive, has a geometric change and belongs to the class called *geometrically-uniform function*[4] by Grossman and Katz [13]. A characteristic of this class of functions is that for each arithmetic progression of arguments, the corresponding sequence of values is a geometric progression, which means that if we take the value of a geometrically-uniform function on the progression t, $t+h$, $t+2h\ldots$, we find the values $x(t)$, $\alpha x(t)$, $\alpha^2 x(t)\ldots$ Those arguments reinforced our change of perspective on the growth function.

This change of perspective seems apparently harmless but in fact it changes the way we look at growth phenomenon. With $x^\star(t)$ the product derivative of $x(t)$ in hand we must first of all explain the change of point of view on the nature of the calculus. For instance, where in the newtonian calculus the fonction with constant dot derivative was the linear one (*i.e.*: if

[4] The functions $e^{\alpha t+\beta}$ and $p^{\alpha t+\beta}$ for p positive constant belong to this class.

$x(t) = at + b$, then $\dot{x}(t) = a$), in the multiplicative calculus it is the power fonction which gives a constant star derivative (*i.e.* : if $x(t) = Ca^t$, then $x^\star(t) = a$). We can also see that in additive calculus if $x(t) = at+b$ then $x(t+1) = x(t)+\dot{x}(t)$ when in the multiplicative calculus if $x(t) = Ca^t$ then $x(t+1) = x(t)\, x^\star(t)$.

Of course, the newtonian and non-newtonian calculus are not completely separated. In fact, we have :

$$\begin{aligned} x^\star(t) &= \lim_{h\to 0}\left(\frac{x(t+h)}{x(t)}\right)^{\frac{1}{h}} = \lim_{h\to 0}\left(1+\frac{x(t+h)-x(t)}{x(t)}\right)^{\left(\frac{x(t)}{x(t+h)-x(t)}\cdot\frac{x(t+h)-x(t)}{h}\cdot\frac{1}{x(t)}\right)} \\ &= \lim_{h\to 0}\left(\left(1+\frac{x(t+h)-x(t)}{x(t)}\right)^{\left(\frac{x(t)}{x(t+h)-x(t)}\right)}\right)^{\left(\frac{x(t+h)-x(t)}{h}\cdot\frac{1}{x(t)}\right)} \\ &= e^{\frac{\dot{x}(t)}{x(t)}} = e^{(\ln\circ x)'(t)} \end{aligned}$$

for $(\ln\circ x)(t) = \ln(x(t))$. If the second order derivative of x at t exists, then by substitution, one will find that :

$$x^{\star\star}(t) = e^{(\ln\circ x^\star)'(t)} = e^{(\ln\circ x)''(t)}$$

We can see that $(\ln\circ x)''$ exists because $x''(t)$ exists. If we repeat n times this procedure, we can conclude that if $x(t)$ is a positive function and $x^{(n)}(t)$ exists,

$$x^{\star(n)}(t) = e^{(\ln\circ x)^{(n)}(t)}$$

We must note that this formula includes the case $n = 0$ as well because[5] :

$$x(t) = e^{(\ln\circ x)(t)}$$

It must signaled that in multiplicative calculus **differentiability* imply continuity but continuity doesn't imply **differentiability*. Secondly the **derivative* of a positive constant function is 1 and to fix some results one has the Table 1.

4 Non-newtonian Euler's formula for homogeneous functions

As an exemple of application of the non-newtonian calculus, we have shown what happen to the celebrated Euler's formula for homogeneous func-

[5]As we has already signaled, for a complete account of the product calculus we must consult Bashirov et al. [2] but it may be useful to begin by Stanley [19].

tions[6], *i.e.* : if we take an homogeneous function of degree r, which means $f(\mu x_1, \ldots, \mu x_n) = \mu^r f(x_1, \ldots, x_n)$ we have

$$rf(x_1, \ldots, x_n) = x_1 f'_{x_1}(x_1, \ldots, x_n) + \cdots + x_n f'_{x_n}(x_1, \ldots, x_n)$$

which is equivalent with

$$r = \frac{x_1 f'_{x_1}(x_1, \ldots, x_n)}{f(x_1, \ldots, x_n)} + \cdots + \frac{x_n f'_{x_n}(x_1, \ldots, x_n)}{f(x_1, \ldots, x_n)}$$

and by exponentiation, we obtain :

$$e^r = \left(e^{\frac{f'_{x_1}}{f}}\right)^{x_1} \cdots \left(e^{\frac{f'_{x_n}}{f}}\right)^{x_n}$$

We introduce the **partial derivative* of f as being

$$f^\star_{x_1} = e^{\frac{f'_{x_1}}{f}}$$

by making a parallel with the definition of the * derivative of f:

$$f^\star = e^{\frac{f'}{f}}$$

We obtain the non-newtonian Euler's formula for homogeneous functions:

$$\left(f^\star_{x_1}\right)^{x_1} \cdots \left(f^\star_{x_1}\right)^{x_n} = e^r$$

In the case of the Cobb-Douglas function, we have :

$$F(K, L) = AK^\alpha L^\beta$$

the non-newtonian Euler's formula is :

$$\left(e^{\frac{\alpha}{K}}\right)^K \left(e^{\frac{\beta}{L}}\right)^L = e^r \iff e^{\alpha+\beta} = e^r$$

in such a way that if there are constant return to scale (the function is homogeneous of degree one), $\alpha + \beta = 1 = r$.

[6]This approach was also developed by Córdova-Lepe [6]. But it's results was given for another definition of multiplicative derivative for positive functions :

$$Qf(x_0) = \lim_{h \to 1} \left(\frac{f(x_0 h)}{f(x_0)}\right)^{\frac{1}{\ln(h)}}, \quad f :]0, \infty[\to]0, \infty[, \quad x_0 \in]0, \infty[, \quad \text{if this limit exists.}$$

5 The Solow-Swan exogenous growth model

In view of what has been explain in the precedent section, one can try to reconstruct the Solow-Swan exogenous growth model. If there is a model know by all macro-economists, it is certainly this one, but it is necessary to remember how it works[7]. We have :

$$\begin{cases} I(t) = S(t) & \text{Equilibrium of the good \& services market} \\ S(t) = sY(t) & \text{Saving function} \\ Y(t) = F(K(t), L(t)) & \text{Production function homogeneous of first degree} \\ \dot{K}(t) = I(t) - \delta K(t) & \text{The increase in the capital is equal to the} \\ & \text{investissement less the obsolescence} \\ \dot{L}(t)/L(t) = n & \text{The constant of growth of the labor force} \end{cases}$$

for I the investment, S the savings, Y the production, K the capital, L the labor force, δ the rate of obsolescence of the capital and n the constant rate of growth of the labor force. We know that if we define $k(t)$ as the capital by head — *i.e.* : $k(t) = K(t)/L(t)$ —, we will find that his accumulation is given by the celebrated ordinary differential equation :

$$\dot{k}(t) = sf(k(t)) - (n + \delta)k(t), \quad k(0) = k_0$$

which can be further developed in postulating for instance that the production function is a Cobb-Douglas one $Y(t) = AK(t)^{\alpha}L(t)^{1-\alpha}$. This gives Bernoulli first order ODE :

$$\dot{k}(t) = sAk^{\alpha}(t) - (n + \delta)k(t), \quad k(0) = k_0$$

whose solution is given by :

$$k(t) = \left[\left[k(0)^{1-\alpha} - \frac{s}{\delta + n}\right] e^{-(1-\alpha)(\delta+n)t} + \frac{s}{\delta + n}\right]^{\frac{1}{1-\alpha}}$$

Since the production fonction is homogeneous of degree one, one knows, by one of the most used Euler theorems, that if the representative enterprise remunerate its factors to the marginal product ($F_K = r/p$ and $F_L = w/p$), the product is completely affected between the two type of costs, *i.e.*:

$$Y = F_K K + F_L L = \frac{r}{p}K + \frac{w}{p}L$$

It is know that this conduct to the fact that the profit by head $\pi = \Pi/L$ could be written :

[7]We must apologize to present here what is elementary text book economics, but we think that, in order to contrast our results with the standard approach this is mandatory.

$$\pi = pf(k) - rk - w$$

which on its turn implies for a maximising firm that $f'(k) = r/p$ and that $\pi = 0$ or equivalently

$$f(k) = \frac{r}{p}k + w$$

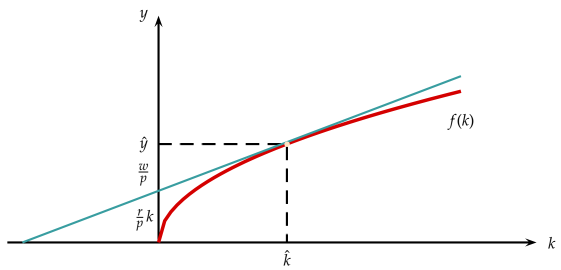

Figure 2: The repartition of the product in the Solow-Swan model

All those informations are subsumed in the Figure 2. We must also stress that in the long term when all capital adjustment has been done, that is to say that, in equilibrium, we have :

$$\dot{k}(t) = 0 \iff sf(\overline{k}) = (n + \delta)\overline{k}$$

which says simply that savings $sf(\overline{k})$ must equal the capital taken into account the constant growth of the labor force and to replace the obsolete one. In the steady state, output per worker is constant but total output increases at the rate n than the labor force. One must notice that obviously we obtain the same results if we decide to use the *derivative to calculate marginal productivity.

This is for the newtonian Solow-Swan model. The problem one face now is to construct the non-newtonian equivalent approach.

On Figure 3, we can see that if savings are greater than $(n+\delta)k$, the capital by workers will tend to rise and, vice versa, if savings are lower than $(n+\delta)k$, it will tend to fall.

Now suppose that with Phelps [17] — see also Burmeister and Dobell [3] — , we want to see if, in the stationary state, there is a better saving rate

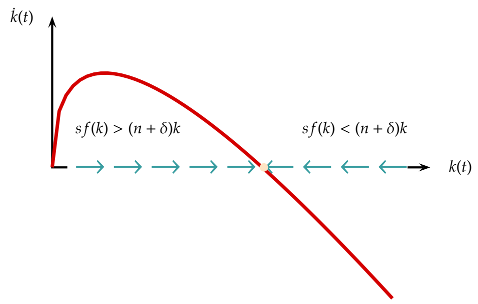

Figure 3: PHASE DIAGRAM OF CAPITAL ACCUMULATION in the Solow model

among all the feasible saving rates. That is to say, we are searching for a saving rate which maximize the consumption by worker in the steady state, *i.e.* :

$$\hat{s} = \text{argmax}_{\{s\}} \{c = (1-s)f(k) | sf(k) = (n+\delta)k\}$$

¿From the constraint it comes that $s = (n+\delta)k/f(k)$ in such a way that the consumption by worker is given by :

$$c = f(k) - (n+\delta)k \implies f'(k) = (n+\delta) \implies \hat{s} = \frac{f'(k)k}{f(k)}$$

In other world, we learned that, if we ignore initial conditions and simply chose the collective saving rate associated with the steady growth path we must equal it to the share of the profit $f'(k)k$ in the product.

6 The non-newtonian Solow-Swan model

If we look carefully to the model, it appears that there are two differential equation that can be rethought to construct a new approach : the equation which described the accumulation of the capital and the equation which describe the growth of the labor force. According to this first one, it's seems natural to postulate for, $0 < \theta < 1$ that :

$$K^\star(t) = \frac{I(t)}{\delta K(t)}$$

Here we must insist that this equation is truly a balance equation even if it doesn't look as such. This come from the fact that double-entry bookkeeping is generally seen as additive but it can also be defined in a multiplicative way[8].

Such an equation described the capital accumulation in a way that for a very short period of time the ratio between the capital of the opening time and the capital of the closing time is raised by the investment but deflated by the obsolescence. In what concern the labor force, according to the mathematical presentation of the product calculus, if one desire not to change the spirit of the model, we must have :

$$L^\star(t) = n$$

In such a way, the non-newtonian Solow-Swan model is now described by the equations :

$$\begin{cases} I(t) = S(t) & \text{Equilibrium of the good \& services market} \\ S(t) = sY(t) & \text{Saving function} \\ Y(t) = F(K(t), L(t)) & \text{Production function homogeneous of first decree} \\ K^\star(t) = \dfrac{I(t)}{\delta K(t)} & \text{The increase in the capital equals the ratio} \\ & \text{between investment and obsolescence} \\ L^\star(t) = n & \text{Growth of the labor force} \end{cases}$$

In order to simplify the presentation we will omit the t argument in the following calculus. So we start with the *derivative of k, and we use :

$$k^\star = \left(\frac{K}{L}\right)^\star = \frac{K^\star}{L^\star}$$

As usual, let $K^\star = I/\delta K$ be transformed on values by capita :

$$K^\star = \frac{\dfrac{I}{L}}{\delta \dfrac{K}{L}} = \frac{\dfrac{sY}{L}}{\delta k} = \frac{sf(k)}{\delta k}$$

which gives the following non-newtonian neoclassical growth differential equation :

$$k^\star(t) = \frac{sf(k(t))}{\delta n k(t)}, \quad k(0) = k_0$$

[8]Surfing on the work of Ellerman [9], we have written a companion paper to explain why the Pacioli-group — it's the name Ellerman has given to the standard or newtonian double entry bookkeeping — is a natural locked-in and why, what we have called the Ellerman-group has a few chances to concurrence it — see Filip and Piatecki [10].

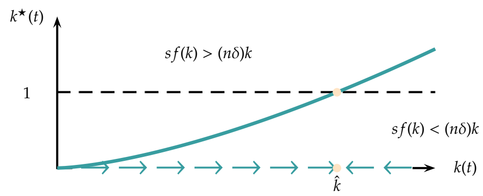

Figure 4: Phase diagram of capital accumulation in the non-newtonian Solow-Swan model

On the stationary state we have $k^\star = 1$ which conducts to :

$$sf(\hat{k}) = (\delta n)\hat{k}$$

We can see that, in comparison with the steady state of the newtonian Solow-Swan model, the difference comes from the apparition of of δn instead of $(\delta + n)$. So the output per worker is constant but it must equalize the multiplicative effect of the increase in the population and the obsolescence.

If we use the definition of $k^\star$, we could come back to a newtonian ODE:

$$e^{\left(\frac{\dot{k}(t)}{k(t)}\right)} = \frac{sf(k(t))}{\delta n k(t)}$$

or

$$\dot{k}(t) = k(t)\left(\ln\left(\frac{s}{\delta n}\right) + \ln\left(\frac{f(k(t))}{k(t)}\right)\right), \quad k(0) = k_0$$

Such an equation gives an interesting alternative to the standard Solow-Swan ODE. If $f(k) = Ak^\alpha$, we find :

$$\dot{k}(t) = k(t)\left(\ln\left(\frac{sA}{\delta n}\right) + (\alpha - 1)\ln(k(t))\right), \quad k(0) = k_0$$

We can see that the non-newtonian Solow-Swan model is not in nature so distinct from the newtonian one. For the same vector of parameters, the non-newtonian growth model shows an amplification of the growth phenomenon.

In the case of a Cobb-Douglas function, we know that the Solow-Swan model conduct to a Bernoulli ODE, which has a known solution. In the non-newtonian approach, we found, with a little Mathematica 6 help, that

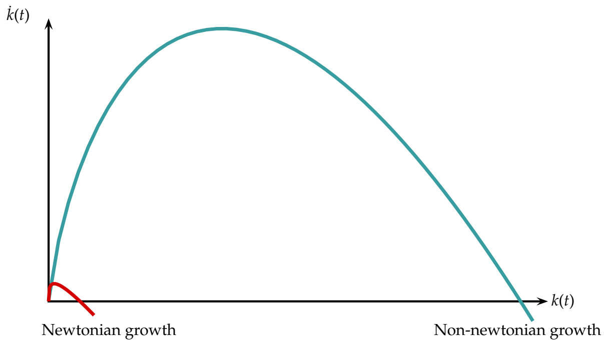

Figure 5: Scale comparison between newtonian and non-newtonian growth

$$y(t) = e^{\frac{1}{\alpha-1}\left(e^{(\alpha-1)(t+C)} + \ln\left(\frac{n\delta}{As}\right)\right)}$$

where

$$C = \ln\left(\ln\left(\frac{Ask_0^{\alpha-1}}{n\delta}\right)^{\frac{1}{\alpha-1}}\right)$$

was obtained by the initial condition $k(0) = k_0$.

From a general point of view, we can also analyse the conditions under which locally $k(t)$ converges to $\hat{k}$. Restricted to the first order the Taylor series of the right side of $\dot{k}(t)$ in the neighborhood of $\hat{k}$ is given by :

$$\hat{k}\left(\ln\left(\frac{s}{n\delta}\right) + \ln\left(\frac{f(\hat{k})}{\hat{k}}\right)\right) + \left(-1 + \ln\left(\frac{s}{n\delta}\right) + \ln\left(\frac{f(\hat{k})}{\hat{k}}\right) + \frac{\hat{k}f'(\hat{k})}{f(\hat{k})}\right)(k - \hat{k}) + O(k - \hat{k})^2$$

Since the first term is equal to 0, if we write $x = k(t) - \hat{k}$, we obtain $\dot{x}(t) = \dot{k}(t)$, in such a way that :

$$\dot{x}(t) = \left(\frac{\hat{k}f'(\hat{k})}{f(\hat{k})} - 1\right)x(t) = \frac{-1}{f(\hat{k})}\left(f(\hat{k}) - \hat{k}f'(\hat{k})\right)x(t)$$

And as $f(\hat{k}) \geq 0$, a necessary and suffisant condition for the growth to converge toward the long term equilibrium is that $f(\hat{k}) > \hat{k}f'(\hat{k})$ which is always fulfilled due to the concavity of the neoclassical production function.

To finish, let us now look at what occurs to the golden rule under the non-newtonian hypothesis. In the steady state, we are searching for a saving rate which maximize the share of the consumption by worker in the income by worker or, equivalently, which minimise the share of the saving by workers , *i.e.* :

$$\hat{s} = \text{argmin}_{\{s\}} \left\{ c = \frac{sf(k)}{f(k)} \middle| sf(k) = n\delta k \right\}$$

From the constraint, it comes that $s = n\delta k/f(k)$ in such a way that , at the minimum :

$$1 = \frac{(n\delta k)^\star}{(f(k))^\star} \iff (f(k))^\star = e^{\frac{1}{k}} \iff e^{\frac{f'(k)}{f(k)}} = e^{\frac{1}{k}} \iff \frac{kf'(k)}{f(k)} = 1$$

In other world, we learned that, if we ignore initial conditions and simply chose the collective saving, we must choose it in such a way that the income by worker elasticity must be unitary. This is a unexpected golden rule which must be contrasted with the Phelps one.

7 Conclusion

In this paper, we have constructed a non-newtonian Solow-Swan growth model on the basis the multiplicative calculus. As newtonian calculus is a locked-in, we doesn't expect that it will be adopted by the major part of the economic community. But, it may appears as an alternative to the standard model of exogenous growth which, we expect, could be useful in the growth debate.

We must underscore that to discover that there was a non-newtonian way to look to differential equation has been a great surprise for us. It opens the question to know if there are major fields of economic analysis which can be profoundly re-thought in the light of this discovery. The field of welfare economics seems to be a natural candidate because as the newtonian analysis has only a way to defined sums on a continuous support, it has mainly been developed under the benthamian hypothesis of utilitarianism. We think there are other ways.

ACKNOWLEDGEMENTS.

The first author aknowledges the support of her work by CNCSIS–UEFISCSU, project number PNII IDEI 2366/2008. The authors thank the Non-Newtonian Calculus page which has cited their working paper since

2010 and David Ellerman for his warm encouragements to continue our work.

References

[1] Arthur, B. [1994]. *Increasing Returns and Path Dependence in the Economy*, The University of Michigan Press.

[2] Bashirov, A., Kurpinar, E. and Özyapici, A. [2008]. Multiplicative calculus and its applications, *Journal of Mathematical Analysis and Applications* **337**: 36–48.

[3] Burmeister, E. and Dobell, A. [1972]. *Mathematical Theories of Economic Growth*, The MacMillan Company.

[4] Cajori, F. [1909]. *A History of Mathematics*, The MacMillan Company.

[5] Campbell, D. [1999]. Multiplicative calculus and student projects, *Primus* **IX**: 327–333.

[6] Córdova-Lepe, F. [2006]. The multiplicative derivative as a measure of elasticity in economics, *TMAT Revista Latinoamericana de Ciencias e Ingeneríria* **2**(3).

[7] Cowan, R. and Gunby, P. [1996]. Sprayed to death: path dependence, lock-in and pest control strategies, *The Economic Journal* **106**(436): 521–542.

[8] David, P. [1985]. Clio and the economics of qwerty, *The American EconomicReview* **75**(2, Papers and Proceedings of the Ninety-Seventh Annual Meeting of the American Economic Association): 332–337.

[9] Ellerman, D. [1985]. The mathematics of double entry bookkeeping, *Mathematics Magazine* **58**(4): 226–233.

[10] Filip, D. and Piatecki, C. [2013]. In defense of a non-newtonian economic analysis, *Technical report*.

[11] Grabiner, J. [1983]. Who gave you the epsilon? Cauchy and the origins of rigorous calculus, *The American Mathematical Monthly* **90**(3): 185194.

[12] Grossman, M. [1979]. An introduction to non-newtonian calculus, *International Journal of Mathematical Education in Science and Technology* **10**(4): 524–528.

[13] Grossman, M. and Katz, R. [1972]. *Non-Newtonian Calculus*, Lee Press.

[14] Hataway, O. [2001]. Path dependance in the law: The course and pattern of legal change in a Common Law system, *Iowa Law Review* **86**.

[15] Malthus, T. [1798]. *An Essay on the Principle of Population, as it Affects the Future Improvement of Society with Remarks on the Speculations of Mr. Godwin, M. Condorcet, and Other Writers*, Oxford World's Classic.

[16] North, D. [1990]. *Institutions, Institutional Change, and Economic Performance*, Cambridge University Press.

[17] Phelps, E. [1965]. Second essay on the golden rule of accumulation, *The American Economic Review* **LV**(4): 793–814.

[18] Solow, R. [1956]. A contribution to the theory of economic growth, *Quarterly Journal of Economics* **70**(1): 65–94.

[19] Stanley, D. [1999]. A multiplicative calculus, *Primus* **9**(4): 310–326.

[20] Swan, T. [1956]. Economic growth and capital accumulation, *Economic Record* **XXXII**: 334–361.

[21] Verhulst, P. [1838]. Notice sur la loi que la population poursuit dans dans son accroissement, *Correspondance Mathématique et Physique* **10**: 131–121.

[22] Volterra, V. [1887]. Sui fondamenti della teoria delle equazioni differenziali lineari, *Technical Report VI*, Memorie della Societa Italiana della Scienze.

	Derivative Rules	Examples	
		$x(t)$	$x^\star(t)$
	$(cx)^\star(t) = (x)^\star(t)$	C	1
Product	$(xy)^\star(t) = x^\star(t)y^\star(t)$	$Ce^{\alpha t}$	e^{α}
Quotient	$\left(\frac{x}{y}\right)^\star(t) = \frac{x^\star(t)}{y^\star(t)}$		
		$Ce^{\sin(t)}$	$e^{\cos(t)}$
		$C\alpha^t$	α
	$(x^y)^\star(t) = x^\star(t)^{y(t)}x(t)^{y'(t)}$	Ct	$e^{\frac{1}{t}}$
Chain rule	$(x \circ y)^\star(t) = x^\star(y(t))^{y'(t)}$	$\alpha t + \beta$	$e^{\frac{\alpha}{\alpha t+\beta}}$
Sum rule	$(x+y)^\star(t) = x^\star(t)^{\frac{x(t)}{x(t)+y(t)}}\, y^\star(t)^{\frac{y(t)}{x(t)+y(t)}}$	Ct^{α}	$e^{\frac{\alpha}{t}}$
		$C\ln(t)$	$e^{\frac{1}{t\ln(t)}}$
		$C\ln(x(t))$	$[x^\star(t)]^{\frac{1}{\ln(x(t))}}$
		$C\sin(t)$	$e^{\cot(t)}$
		$C\cos(t)$	$e^{\tan(t)}$

Table 1: SOME EXEMPLES OF PRODUCT CALCULUS