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Reduction of Higher Order Linear Ordinary Differential Equations into the Second Order and Integral Evaluation of Exact Solutions

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Abstract

Higher order linear differential equations with arbitrary order and variable coefficients are reduced in this work. The method is based on the decomposition of their coefficients and the approach reduces the order until second order equation is produced. The method to find closed-form solutions to the second order equation is then developed. The solution for the second order ODE is produced by rearranging its coefficients. Exact integral evaluation is also conducted to complete the solutions.

Keywords: Higher order linear ordinary differential equations, exact integral evaluation, reduction of order, decomposition of variable coefficients.

MSC number (2010): 34A05, 34A25, 34A30.

1. Introduction

It is well-known that the well-posed problem for the linear differential equations has been settled and completed by means of functional analysis [1]. However, the concepts will not be very useful until the explicit solutions are produced. They are capable to describe the detail features of the systems [2,3]. They may also help to extend the existence, uniqueness and regularity properties of the solutions which are obtained from qualitative analysis [4].

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Therefore, methods for solving linear differential equations with variable coefficients are important from both physical and mathematical point of views [5]. Especially for the second order ordinary differential equations, with nonhomogenous physical properties, such as in waves propagation in non-uniform media and vibration waves with anisotropic physical properties. Since that specific problem attracts many mathematicians and physicists, the methods to obtain exact and approximate solutions for second order equation are tackled systematically and some interesting results are produced [6]. One case is the method of differential transfer matrix to handle some physical problems which is computationally milder than the previous analytic methods and the method is also applied to the higher order ODEs [7]. Also some approximate methods can be extended to handle nonlinear equations [8,9]. Despite the concentrated research and reports on the problem, the closed-form solutions for the higher order and second order ODEs with variable coefficients remain one of the important area of differential equations [10]. Even it is recently claimed that the problem is not solvable in general case [11].

In this work, the method for obtaining exact solutions to the second order equations is conducted by rearranging the coefficients. The solution of the second order equation will be implemented as a basis for tackling the higher order equations. The coefficients of the equations are decomposed in order to reduce their order. The reduction is continued until the second order equation is produced and solved. The explicit expression then can be determined by the proposed exact integral evaluation in order to complete solutions. Finally, we give ilustrations of integral evaluation by examples.

2. Solutions for Second Order Differential Equations

Since the second order differential equation can be transformed into the Riccati class, we begin from the following statement,

Theorem 1*: Consider the second order linear ODE with variable coefficients,*

$$y_{xx} + a_1 y_x + a_2 y = 0$$

The coefficients a_1 *and* a_2 *can be split into new functions,* $f_1, f_2, f_3, f_4, f_5, f_6$ *and* α*. By determining the new functions* $f_1, f_2, f_3, f_4, f_5, f_6$ *and* α*, the closed-form solution is obtained as,*

$$y = \frac{1}{f_3 f_6}\left(\int_x a_2 dx\right)^{-1}\left[C_6 \int_x f_3^2 f_6^2 \left(\int_x a_2 dx\right)^2 e^{-\int_x a_1 dx} dx + C_7\right]$$

where $$f_3 = \left[C_1 \int_x \left(\int_x a_2 dx\right)^2 e^{-\int_x a_1 dx} dx + C_2\right]^{-1}$$ *and*

$$f_6 = \left\{C_4 \int_x \left[C_1 \int_x \left(\int_x a_2 dx\right)^2 e^{-\int_x a_1 dx} dx + C_2\right]^{-2} dx + C_5\right\}^{-1}.$$

***Proof**:* The above equation can be rewritten as,

$$\frac{1}{\alpha}(\alpha y_x)_x + \left(a_1 - \frac{\alpha_x}{\alpha}\right) y_x + a_2 y = 0 \tag{1a}$$

Suppose that, $\left(a_1 - \frac{\alpha_x}{\alpha}\right)_x = a_2$ to produce, $\alpha = C_1 e^{\int_x \left(a_1 - \int_x a_2 dx\right) dx}$. The above equation can be rearranged as,

$$\chi_{xx} + a_3 \chi_x + a_4 \chi = 0 \tag{1b}$$

with, $\left(a_1 - \frac{\alpha_x}{\alpha}\right) y = \chi$, $a_3 = \left(a_1 - \frac{\alpha_x}{\alpha}\right)\left\{\frac{1}{\alpha}\left[\alpha\left(a_1 - \frac{\alpha_x}{\alpha}\right)^{-1}\right]_x + \left[\left(a_1 - \frac{\alpha_x}{\alpha}\right)^{-1}\right]_x + 1\right\}$ and

$a_4 = \left(a_1 - \frac{\alpha_x}{\alpha}\right)\left\{\alpha\left[\left(a_1 - \frac{\alpha_x}{\alpha}\right)^{-1}\right]_x\right\}_x$. Equation (1b) can be rearranged as,

$$\left(f_1 \chi_x\right)_x + \left(a_3 f_1 - f_{1x}\right)\chi_x + a_4 f_1 \chi = 0 \tag{1c}$$

Set, $a_4 f_1 = f_2 + \left(a_3 f_1 - f_{1x}\right)\frac{f_{3x}}{f_3}$ to get,

$$\left(f_1 \chi_x\right)_x + \frac{\left(a_3 f_1 - f_{1x}\right)}{f_3}\left(f_3 \chi\right)_x + f_2 \chi = 0 \tag{1d}$$

Let, $f_3 \chi = \varphi$, equation (1e) will become,

$$\left[\frac{f_1}{f_3}\varphi_x + f_1\left(\frac{1}{f_3}\right)_x \varphi\right]_x + \frac{\left(a_3 f_1 - f_{1x}\right)}{f_3}\varphi_x + \frac{f_2}{f_3}\varphi = 0 \text{ or } \varphi_{xx} + a_5 \varphi_x + a_6 \varphi = 0 \tag{2a}$$

with , $a_5 = 2 f_3 \left(\frac{1}{f_3}\right)_x + a_3$ and $a_6 = \frac{f_{1x}}{f_1} f_3 \left(\frac{1}{f_3}\right)_x + f_3 \left(\frac{1}{f_3}\right)_{xx} + \frac{f_2}{f_1}$. Repeat the procedure (1c – d) to produce,

$$\left[\frac{f_4}{f_6}\vartheta_x + f_4\left(\frac{1}{f_6}\right)_x \vartheta\right]_x + \frac{\left(a_5 f_4 - f_{4x}\right)}{f_6}\vartheta_x + \frac{f_5}{f_6}\vartheta = 0 \text{ or } \vartheta_{xx} + a_7 \vartheta_x + a_8 \vartheta = 0 \tag{2b}$$

where the relations, $a_6 f_4 = f_5 + (a_5 f_4 - f_{4x})\frac{f_{6x}}{f_6}$, $a_7 = 2f_6\left(\frac{1}{f_6}\right)_x + a_5$, $a_8 = \frac{f_{4x}}{f_4} f_6\left(\frac{1}{f_6}\right)_x + f_6\left(\frac{1}{f_6}\right)_{xx} + \frac{f_5}{f_4}$ and $f_6\varphi = \vartheta$ are hold. Let, $a_8 = \frac{f_{4x}}{f_4} f_6\left(\frac{1}{f_6}\right)_x + f_6\left(\frac{1}{f_6}\right)_{xx} + \frac{f_5}{f_4} = 0$, the solution for f_4 is,

$$f_4 = -\left(\frac{1}{f_6}\right)_x^{-1}\left(\int_x \frac{f_5}{f_6}dx + C\right) \tag{2c}$$

Substituting the above equation into, $a_6 f_4 = f_5 + (a_5 f_4 - f_{4x})\frac{f_{6x}}{f_6}$, to get,

$$-a_6\left(\frac{1}{f_6}\right)_x^{-1}\left(\int_x \frac{f_5}{f_6}dx + C\right) = f_5 - a_5\frac{f_{6x}}{f_6}\left(\frac{1}{f_6}\right)_x^{-1}\left(\int_x \frac{f_5}{f_6}dx + C\right) + \left(\frac{1}{f_6}\right)_x^{-1}\frac{f_5}{f_6}\frac{f_{6x}}{f_6} - \left(\frac{1}{f_6}\right)_{xx}\left(\frac{1}{f_6}\right)_x^{-2}\frac{f_{6x}}{f_6}\left(\int_x \frac{f_5}{f_6}dx + C\right)$$

or

Take, $a_6 = \frac{f_5}{f_1}$, the solution for f_5 can be obtained as,

$$f_5 = \left[a_5\frac{f_{6x}}{f_6} + \left(\frac{1}{f_6}\right)_{xx}\left(\frac{1}{f_6}\right)_x^{-1}\frac{f_{6x}}{f_6}\right] f_1 \tag{2d}$$

Recall the definition of a_5 and a_6, substitute $a_4 f_1 = f_2 + (a_3 f_1 - f_{1x})\frac{f_{3x}}{f_3}$ and equating with (2d) to form,

$$a_6 = \frac{f_{1x}}{f_1} f_3\left(\frac{1}{f_3}\right)_x + f_3\left(\frac{1}{f_3}\right)_{xx} + \frac{f_2}{f_1} = a_5\frac{f_{6x}}{f_6} + \left(\frac{1}{f_6}\right)_{xx}\left(\frac{1}{f_6}\right)_x^{-1}\frac{f_{6x}}{f_6} \text{ or}$$

$$f_3\left(\frac{1}{f_3}\right)_{xx} + a_4 - a_3\frac{f_{3x}}{f_3} = \frac{1}{f_1}\left[\left(a_3 - 2\frac{f_{3x}}{f_3}\right)\frac{f_{6x}}{f_6} - f_6\left(\frac{1}{f_6}\right)_{xx}\right] \tag{3a}$$

Let, $a_4 = \frac{1}{f_1}a_3\frac{f_{6x}}{f_6}$, the above equation can be written as,

$$f_3\left(\frac{1}{f_3}\right)_{xx} - a_3\frac{f_{3x}}{f_3} = \frac{1}{f_1}\left[-2\frac{f_{3x}}{f_3}\frac{f_{6x}}{f_6} - f_6\left(\frac{1}{f_6}\right)_{xx}\right] \tag{3b}$$

Suppose that, $f_3\left(\frac{1}{f_3}\right)_{xx} - a_3\frac{f_{3x}}{f_3} = 0$, the solution for f_3 and f_6 are then,

$$f_3 = \left(C_1\int_x e^{-\int_x a_3 dx}dx + C_2\right)^{-1} \text{ and } f_6 = \left[C_4\int_x\left(C_1\int_x e^{-\int_x a_3 dx}dx + C_2\right)^{-2}dx + C_5\right]^{-1} \tag{3c}$$

Note that, $f_2 = (f_{1x} - a_3 f_1)\frac{f_{3x}}{f_3} + a_4 f_1$, with $f_1 = \frac{a_3}{a_4}\frac{f_{6x}}{f_6}$ and f_3 is expressed by (3c).

Therefore, equation (2b) becomes,

$$\vartheta_{xx} + a_7 \vartheta_x = 0 \text{ or } \vartheta_{xx} + \left(a_5 - 2\frac{f_{6x}}{f_6} \right) \vartheta_x = 0 \qquad (4a)$$

The solution for (4a) is then,

$$\vartheta = C_6 \int_x e^{\int_x \left(2\frac{f_{6x}}{f_6} - a_5 \right) dx} dx + C_7 \text{ or } \vartheta = C_6 \int_x f_3^2 f_6^2 e^{-\int_x a_3 dx} dx + C_7 \qquad (4b)$$

The solution for y is defined as,

$$y = \frac{1}{f_3 f_6} \left(a_1 - \frac{\alpha_x}{\alpha} \right)^{-1} \left(C_6 \int_x f_3^2 f_6^2 e^{-\int_x a_3 dx} dx + C_7 \right) =$$
$$\frac{1}{f_3 f_6} \left(\int_x a_2 dx \right)^{-1} \left[C_6 \int_x f_3^2 f_6^2 \left(\int_x a_2 dx \right)^2 e^{-\int_x a_1 dx} dx + C_7 \right] \qquad (4c)$$

where $f_3 = \left[C_1 \int_x \left(\int_x a_2 dx \right)^2 e^{-\int_x a_1 dx} dx + C_2 \right]^{-1}$ and

$f_6 = \left\{ C_4 \int_x \left[C_1 \int_x \left(\int_x a_2 dx \right)^2 e^{-\int_x a_1 dx} dx + C_2 \right]^{-2} dx + C_5 \right\}^{-1}$. This proves theorem 1.

3. Cases of Order Reduction

Consider a non homogenous third order linear differential equation with variable coefficients below,

$$y_{xxx} + a_1 y_{xx} + a_2 y_x + a_3 y = a_4 \qquad (5a)$$

Lemma 1*: Equation (5a) is reducible into second order equation and has closed-form exact solutions.*

Proof*:* Let,

$$a_1 = b_1 + \frac{a_{5x}}{a_5} \qquad (5b)$$

Then, the equation can be rewritten in the following form,

$$\frac{1}{a_5} \left(a_5 y_{xx} \right)_x + b_1 y_{xx} + a_2 y_x + a_3 y = a_4$$

Set,

$$a_2 = b_2 + b_1 \frac{a_{6x}}{a_6} \qquad (5c)$$

Thus, the following relation is obtained,

$$\frac{1}{a_5} \left(a_5 y_{xx} \right)_x + \frac{b_1}{a_6} \left(a_6 y_x \right)_x + b_2 y_x + a_3 y = a_4$$

Multiply by an arbitrary function α to generate [8],

$$\frac{\alpha}{a_5}\left(a_5 y_{xx}\right)_x + \frac{\alpha b_1}{a_6}\left(a_6 y_x\right)_x + \alpha b_2 y_x + \alpha a_3 y = \alpha a_4 \tag{5d}$$

Suppose that the following expression is satisfied,

$$\alpha_x b_2 = \alpha a_3 \text{ then, } \alpha = C_1 e^{\int_x \frac{a_3}{b_2} dx} \tag{5e}$$

Let $C_1 = 1$, equation (5d) is rewritten as,

$$\frac{\alpha}{a_5}\left(a_5 y_{xx}\right)_x + \frac{\alpha b_1}{a_6}\left(a_6 y_x\right)_x + b_2\left(e^{\int_x \frac{a_3}{b_2} dx} y\right)_x = \alpha a_4$$

Suppose that,

$$e^{\int_x \frac{a_3}{b_2} dx} y = u \text{, and } y = u e^{-\int_x \frac{a_3}{b_2} dx} \tag{5f}$$

Therefore equation (5d) can be expanded as,

$$\frac{\alpha}{a_5}\left\{a_5\left[u_{xx} e^{-\int_x \frac{a_3}{b_2} dx} + 2u_x\left(-\frac{a_3}{b_2}\right) e^{-\int_x \frac{a_3}{b_2} dx} + u\left(-\frac{a_3}{b_2}\right)^2 e^{-\int_x \frac{a_3}{b_2} dx}\right]\right\}_x + \frac{\alpha b_1}{a_6}\left\{a_6\left[u_x e^{-\int_x \frac{a_3}{b_2} dx} + u\left(-\frac{a_3}{b_2}\right) e^{-\int_x \frac{a_3}{b_2} dx}\right]\right\}_x$$
$$+ b_2 u_x = \alpha a_4$$

Differentiate the above equation once again and set the following relation,

$$\frac{\alpha}{a_5}\left[a_5\left(-\frac{a_3}{b_2}\right)^2 e^{-\int_x \frac{a_3}{b_2} dx}\right]_x + \frac{\alpha b_1}{a_6}\left[a_6\left(-\frac{a_3}{b_2}\right) e^{-\int_x \frac{a_3}{b_2} dx}\right]_x = 0 \tag{6a}$$

Now assume that b_2 is given, then $\frac{a_{6x}}{a_6}$ can be determined from (6a) as,

$$\frac{a_{6x}}{a_6} = \frac{f_6 \frac{a_{5x}}{a_5} + b_1 f_7 + f_8}{b_1 f_9} \tag{6b}$$

Substituting into (5c) to give the expression of b_1 as a function of $\frac{a_{5x}}{a_5}$. Performing the resulting expression into (5b) to generate a_5. Therefore equation (5a) is reduced into,

$$u_{xxx} + a_7 u_{xx} + a_8 u_x = a_9$$

Let, $u_x = v$, thus the above equation be transformed to the second order ODE,

$$v_{xx} + a_7 v_x + a_8 v = a_9 \tag{6c}$$

Then, by the application of theorem 1, equation (6c) is solvable in closed-form. The non homogenous part is covered by taking homogenous solution of (6c) as a particular solution in the following form,

$$u_x = \phi\psi \tag{6d}$$

where ϕ is a particular solution from equation (6c). The solution for ψ is stated as,

$$\psi = \phi\int_x \left(\frac{1}{\phi^2} e^{-\int_x a_7 dx} \int_x e^{\int_x a_7 dx} \phi a_9 dx \right) dx \tag{6e}$$

The combination of (6e) with (6f) and (6d) will produce the final solution. This proves lemma 1.

Lemma 2*: The fourth order linear differential equation,*

$$y_{xxxx} + a_1 y_{xxx} + a_2 y_{xx} + a_3 y_x + a_4 y = a_5$$

is reducible to third and second order equations and has closed-form solutions.

Proof*:* Suppose that,

$$a_1 = b_1 + \frac{a_{5x}}{a_5},\ a_2 = b_2 + b_1 \frac{a_{6x}}{a_6} \text{ and } a_3 = b_3 + b_2 \frac{a_{7x}}{a_7} \tag{7a}$$

Therefore the equation become,

$$\frac{1}{a_5}(a_5 y_{xxx})_x + \frac{b_1}{a_6}(a_6 y_{xx})_x + \frac{b_2}{a_7}(a_7 y_x)_x + b_3 y_x + a_4 y = a_5$$

Multiplying by an arbitrary function α to give,

$$\frac{\alpha}{a_5}(a_5 y_{xxx})_x + \frac{\alpha b_1}{a_6}(a_6 y_{xx})_x + \frac{\alpha b_2}{a_7}(a_7 y_x)_x + \alpha b_3 y_x + \alpha a_4 y = a_5 \tag{7b}$$

Let,

$$\alpha_x b_3 = \alpha a_4 \text{ then, } \alpha = C_1 e^{\int_x \frac{a_4}{b_3} dx} \tag{7c}$$

Equation (7c) is transformed as,

$$\frac{\alpha}{a_5}(a_5 y_{xxx})_x + \frac{\alpha b_1}{a_6}(a_6 y_{xx})_x + \frac{\alpha b_2}{a_7}(a_7 y_x)_x + b_3 \left(e^{\int_x \frac{a_4}{b_3} dx} y \right)_x = \alpha a_5$$

Let us assume that,

$$e^{\int_x \frac{a_4}{b_3} dx} y = u, \text{ and } y = u e^{-\int_x \frac{a_4}{b_3} dx} \tag{7d}$$

Expanding equation (7b) as,

$$\frac{\alpha}{a_5}\left\{a_5\left[u_{xxx}e^{-\int_x \frac{a_4}{b_3}dx}+3u_{xx}\left(-\frac{a_4}{b_3}\right)e^{-\int_x \frac{a_4}{b_3}dx}+3u_x\left(-\frac{a_4}{b_3}\right)^2 e^{-\int_x \frac{a_4}{b_3}dx}+u\left(-\frac{a_4}{b_3}\right)^3 e^{-\int_x \frac{a_4}{b_3}dx}\right]\right\}_x+$$

$$\frac{\alpha b_1}{a_6}\left\{a_6\left[u_{xx}e^{-\int_x \frac{a_4}{b_3}dx}+2u_x\left(-\frac{a_4}{b_3}\right)e^{-\int_x \frac{a_4}{b_3}dx}+u\left(-\frac{a_4}{b_3}\right)^2 e^{-\int_x \frac{a_4}{b_3}dx}\right]\right\}_x+\frac{\alpha b_2}{a_7}\left\{a_7\left[u_x e^{-\int_x \frac{a_4}{b_3}dx}+u\left(-\frac{a_4}{b_3}\right)e^{-\int_x \frac{a_4}{b_3}dx}\right]\right\}_x$$

$$+b_3u_x=\alpha a_5$$

Performing the following relation,

$$\frac{\alpha}{a_5}\left[a_5\left(-\frac{a_4}{b_3}\right)^3 e^{-\int_x \frac{a_4}{b_3}dx}\right]_x+\frac{\alpha b_1}{a_6}\left[a_6\left(-\frac{a_4}{b_3}\right)^2 e^{-\int_x \frac{a_4}{b_3}dx}\right]_x+\frac{\alpha b_2}{a_7}\left[a_7\left(-\frac{a_4}{b_3}\right)e^{-\int_x \frac{a_4}{b_3}dx}\right]_x=0 \qquad (8a)$$

Suppose that b_3 and a_7 are given, then $\frac{a_{6x}}{a_6}$ can be determined form (8a) as,

$$\frac{a_{6x}}{a_6}=\frac{f_{10}\frac{a_{5x}}{a_5}+b_1f_{11}+f_{12}}{b_1f_{13}} \qquad (8b)$$

Substituting (8b) into the second relation of (7a) to give b_1 as a function of $\frac{a_{5x}}{a_5}$. The next step is implementing into the first relation of (7a) to produce a_5. Therefore, the fourth order equation is reduced into,

$$u_{xxxx}+a_8u_{xxx}+a_9u_{xx}+a_{10}u_x=a_{11}$$

Let, $u_x=v$, thus the above equation can be transformed to the third order equation,

$$v_{xxx}+a_8v_{xx}+a_9v_x+a_{10}v=a_{11} \qquad (8c)$$

Then, by the application of theorem 1 and lemma 1, equation (8c) will have closed-form solutions. This proves lemma 2.

It is interesting to note that, by induction, the procedure can be applied to any order higher than two and the considered equations are transformed into the second order equations.

Theorem 2*: Higher order linear differential equation is reducible into the second order equation and has closed-form solutions.*

4. Remarks on Integral Evaluation

It is important to note that the integrals which appear in the exact solutions are usually approximated in series form [12]. The solutions consequently are no longer exact. In order to resolve the problem, now the following integral is considered,

$$B=\int_{\xi}\lambda e^{\int_{\xi} f d\xi}d\xi \tag{9a}$$

By setting,

$$B=\int_{\xi}\lambda e^{\int_{\xi} f d\xi}d\xi=(R+Q)\eta e^{\int_{\xi} g d\xi} \tag{9b}$$

Equation (9b) can be differentiated once to give,

$$\lambda e^{\int_{\xi} f d\xi}=\left(R_{\xi}+Q_{\xi}\right)\eta e^{\int_{\xi} g d\xi}+(R+Q)\eta_{\xi}e^{\int_{\xi} g d\xi}+(R+Q)\eta g e^{\int_{\xi} g d\xi}$$

Rearranging the above equation as,

$$R_{\xi}+\left(\frac{\eta_{\xi}}{\eta}+g\right)R=\frac{\lambda}{\eta}e^{\int_{\xi} f-g d\xi}-\left\{Q_{\xi}+\left(\frac{\eta_{\xi}}{\eta}+g\right)Q\right\} \tag{9c}$$

The solution of R is then expressed by,

$$R=\frac{1}{\eta}e^{-\int_{\xi} g d\xi}\left\{\int_{\xi}\eta e^{\int_{\xi} g d\xi}\left[\frac{\lambda}{\eta}e^{\int_{\xi} f-g d\xi}-\left\{Q_{\xi}+\left(\frac{\eta_{\xi}}{\eta}+g\right)Q\right\}\right]d\xi+C_1\right\} \tag{9d}$$

Let,

$$\frac{\lambda}{\eta}e^{\int_{\xi} f-g d\xi}-\left\{Q_{\xi}+\left(\frac{\eta_{\xi}}{\eta}+g\right)Q\right\}=f_{14} \tag{9e}$$

Then, R is evaluated in the following,

$$R=\frac{1}{\eta}e^{-\int_{\xi} g d\xi}\left[\left(\int_{\xi} f_{14}\eta d\xi\right)e^{\int_{\xi} g d\xi}-\int_{\xi}\left(\int_{\xi} f_{14}\eta d\xi\right)g e^{\int_{\xi} g d\xi}d\xi+C_1\right] \tag{9f}$$

Suppose that from equation (9e),

$$\lambda e^{\int_{\xi} f-g d\xi}=C_2$$

where C_2 is also a constant.

The expression for $e^{\int_{\xi} g d\xi}$ is written as,

$$\frac{\lambda}{C_2\eta}e^{\int_{\xi} f d\xi}=e^{\int_{\xi} g d\xi} \tag{10a}$$

Thus, equation (9f) will become,

$$R=\frac{1}{C_2\eta}e^{-\int_\xi g d\xi}\left[\left(\int_\xi f_{14}\eta d\xi\right)\frac{\lambda}{\eta}e^{\int_\xi f d\xi}-\int_\xi\left(\int_\xi f_{14}\eta d\xi\right)\left(\frac{\lambda}{\eta}e^{\int_\xi f d\xi}\right)_\xi d\xi+C_1\right] \tag{10b}$$

Without loss of generality, set $\int_\xi f_{14}\eta d\xi=\ln\left(\frac{\lambda}{\eta}e^{\int_\xi f d\xi}\right)$, and the expression of f_{14} is obtained as,

$$f_{14}=\frac{1}{\eta}\left\{\ln\left(\frac{\lambda}{\eta}e^{\int_\xi f d\xi}\right)\right\}_\xi \tag{10c}$$

The solution for Q is consequently obtained from (9e) as in the following relation,

$$Q=\frac{1}{\eta}e^{-\int_\xi g d\xi}\int_\xi\left(C_2-f_{14}\right)\eta e^{\int_\xi g d\xi}d\xi$$

Substituting (10a) to get,

$$Q=\frac{1}{C_2\eta}e^{-\int_\xi g d\xi}\int_\xi\left(C_2-f_{14}\right)\lambda e^{\int_\xi f d\xi}d\xi \tag{10d}$$

Equations (9b), (10b) and (10d) will give the evaluation as,

$$\int_\xi \lambda e^{\int_\xi f d\xi}d\xi=\left(R+Q\right)\eta e^{\int_\xi g d\xi}=\frac{1}{C_2}\frac{\lambda}{\eta}e^{\int_\xi f d\xi}+\frac{1}{C_2}\int_\xi\left(C_2-f_{14}\right)\lambda e^{\int_\xi f d\xi}d\xi+C_1 \tag{10e}$$

where f_{14} is determined by (10c).

Equation (10e) can be differentiated once and rearranged to be,

$$\int_\xi \lambda e^{\int_\xi f d\xi}d\xi=\int_\xi\frac{1}{f_{14}}\left(\frac{\lambda}{\eta}e^{\int_\xi f d\xi}\right)_\xi d\xi \tag{11a}$$

Now suppose that $\frac{\lambda}{\eta}e^{\int_\xi f d\xi}=L$ and $f_{14}=\frac{1}{\eta}\left\{\ln\left(\frac{\lambda}{\eta}e^{\int_\xi f d\xi}\right)\right\}_\xi=L^n$, with n is an arbitrary constant. The relation of η is then given by,

$$\lambda^n e^{n\int_\xi f d\xi}\eta^{1-n}=-\frac{\eta_\xi}{\eta}+\left(\frac{\lambda_\xi}{\lambda}+f\right) \tag{11b}$$

Let $\eta=\chi^{\frac{1}{1-n}}$, equation (11b) will then produce,

$$\chi_\xi=\left(n-1\right)\lambda^n e^{n\int_\xi f d\xi}\chi^2+\left(1-n\right)\left(\frac{\lambda_\xi}{\lambda}+f\right)\chi \tag{11c}$$

Let $\chi=-\frac{1}{\left(n-1\right)\lambda^n e^{n\int_\xi f d\xi}}\frac{\gamma_\xi}{\gamma}$, the above equation will then become,

$$-\left(\frac{1}{(n-1)\lambda^n e^{n\int_\xi fd\xi}}\right)_\xi \frac{\gamma_\xi}{\gamma} - \left(\frac{1}{(n-1)\lambda^n e^{n\int_\xi fd\xi}}\right)\frac{\gamma_{\xi\xi}}{\gamma} = \left(\frac{\frac{\lambda_\xi}{\lambda}+f}{\lambda^n e^{n\int_\xi fd\xi}}\right)\frac{\gamma_\xi}{\gamma} \tag{11d}$$

Equation (11d) can be rearranged as,

$$\gamma_{\xi\xi} = \left[\frac{\left(\lambda^n e^{n\int_\xi fd\xi}\right)_\xi}{\lambda^n e^{n\int_\xi fd\xi}} + (1-n)\left(\frac{\lambda_\xi}{\lambda}+f\right)\right]\gamma_\xi$$

The solution for γ is

$$\gamma = \int_\xi \lambda e^{\int_\xi fd\xi} d\xi \tag{11e}$$

The solution for η is

$$\eta = \chi^{\frac{1}{1-n}} = \left(-\frac{1}{(n-1)\lambda^n e^{n\int_\xi fd\xi}}\frac{\gamma_\xi}{\gamma}\right)^{\frac{1}{1-n}} \tag{11f}$$

The step is now perfoming the integration of (11a) to give,

$$\int_\xi \lambda e^{\int_\xi fd\xi} d\xi = \frac{1}{1-n}L^{1-n} = \frac{1}{1-n}\left(\frac{\lambda}{\eta}e^{\int_\xi fd\xi}\right)^{1-n} = \frac{1}{1-n}\left[\frac{1}{\eta}\left\{\ln\left(\frac{\lambda}{\eta}e^{\int_\xi fd\xi}\right)\right\}_\xi\right]^{\frac{1-n}{n}} = \frac{1}{1-n}\left[\frac{1}{\eta}\left(f+\frac{\lambda_\xi}{\lambda}-\frac{\eta_\xi}{\eta}\right)\right]^{\frac{1-n}{n}} \tag{12a}$$

Rearranging (12a) and substituting (11f),

$$(1-n)^{\frac{n}{1-n}}\left(\frac{1}{(1-n)\lambda^n e^{n\int_\xi fd\xi}}\frac{\gamma_\xi}{\gamma}\right)^{\frac{2}{1-n}}\left(\int_\xi \lambda e^{\int_\xi fd\xi}d\xi\right)^{\frac{n}{1-n}} =$$

$$\left(f+\frac{\lambda_\xi}{\lambda}\right)\left(\frac{1}{(1-n)\lambda^n e^{n\int_\xi fd\xi}}\frac{\gamma_\xi}{\gamma}\right)^{\frac{1}{1-n}} - \left[\left(\frac{1}{(1-n)\lambda^n e^{n\int_\xi fd\xi}}\frac{\gamma_\xi}{\gamma}\right)^{\frac{1}{1-n}}\right]_\xi \tag{12b}$$

The polynomial equation for $\gamma = \int_\xi \lambda e^{\int_\xi fd\xi} d\xi$ is then,

$$(1-n)^{\frac{1}{1-n}}\left(\frac{1}{(1-n)\lambda^n e^{n\int_\xi f d\xi}}\frac{\gamma_\xi}{\gamma}\right)^{\frac{2}{1-n}}\left(\int_\xi \lambda e^{\int_\xi f d\xi} d\xi\right)^{\frac{n}{1-n}} = (1-n)\left(f+\frac{\lambda_\xi}{\lambda}\right)\left(\frac{1}{(1-n)\lambda^n e^{n\int_\xi f d\xi}}\frac{\gamma_\xi}{\gamma}\right)^{\frac{1}{1-n}} +$$

$$\left[\left(\frac{1}{(1-n)\lambda^n e^{n\int_\xi f d\xi}}\frac{\gamma_\xi}{\gamma}\right)^{\frac{n}{1-n}}\left(-\left(\frac{1}{(1-n)\lambda^n e^{n\int_\xi f d\xi}}\right)_\xi\frac{\gamma_\xi}{\gamma}\right)-\left(\frac{1}{(1-n)\lambda^n e^{n\int_\xi f d\xi}}\frac{\gamma_{\xi\xi}}{\gamma}\right)+\left(\frac{1}{(1-n)\lambda^n e^{n\int_\xi f d\xi}}\left(\frac{\gamma_\xi}{\gamma}\right)^2\right)\right]$$

or

$$(1-n)^{\frac{1}{1-n}} K_1^2\gamma_\xi^2\gamma^n = (1-n)\left(f+\frac{\lambda_\xi}{\lambda}\right)^{1-n} K_1\gamma_\xi\gamma + K_1^n\gamma_\xi^n\left[-\left(K_{1\xi}\gamma_\xi + K_1\gamma_{\xi\xi}\right)\gamma + K_1\gamma_\xi^2\right]^{1-n} \qquad (12c)$$

with $K_1 = \dfrac{1}{(1-n)\lambda^n e^{n\int_\xi f d\xi}}$. Without loss of generality let $n=2$ to get,

$$\left[K_1^2\gamma_\xi^2\left(K_{1\xi}\gamma_\xi + K_1\gamma_{\xi\xi}\right)\right]\gamma^3 + \left[K_1^3\gamma_\xi^4 + \left(f+\frac{\lambda_\xi}{\lambda}\right)^{-1} K_1\gamma_\xi\left(K_{1\xi}\gamma_\xi + K_1\gamma_{\xi\xi}\right)\right]\gamma^2 +$$
$$\left[\left(f+\frac{\lambda_\xi}{\lambda}\right)^{-1} K_1^2\gamma_\xi^3\right]\gamma + K_1^2\gamma_\xi^2 = 0 \qquad (12d)$$

By using the cubic formula,

$$M = \frac{1}{3}\frac{\left[\left(f+\frac{\lambda_\xi}{\lambda}\right)^{-1} K_1^2\gamma_\xi^3\right]}{\left[K_1^2\gamma_\xi^2\left(K_{1\xi}\gamma_\xi + K_1\gamma_{\xi\xi}\right)\right]} - \frac{1}{9}\left\{\frac{\left[K_1^3\gamma_\xi^4 + \left(f+\frac{\lambda_\xi}{\lambda}\right)^{-1} K_1\gamma_\xi\left(K_{1\xi}\gamma_\xi + K_1\gamma_{\xi\xi}\right)\right]}{\left[K_1^2\gamma_\xi^2\left(K_{1\xi}\gamma_\xi + K_1\gamma_{\xi\xi}\right)\right]}\right\}^2, \qquad (12e)$$

$$N = \frac{1}{6}\left\{\frac{\left[\left(f+\frac{\lambda_\xi}{\lambda}\right)^{-1} K_1^2\gamma_\xi^3\right]\left[K_1^3\gamma_\xi^4 + \left(f+\frac{\lambda_\xi}{\lambda}\right)^{-1} K_1\gamma_\xi\left(K_{1\xi}\gamma_\xi + K_1\gamma_{\xi\xi}\right)\right]}{\left[K_1^2\gamma_\xi^2\left(K_{1\xi}\gamma_\xi + K_1\gamma_{\xi\xi}\right)\right]^2} - \frac{3K_1^2\gamma_\xi^2}{\left[K_1^2\gamma_\xi^2\left(K_{1\xi}\gamma_\xi + K_1\gamma_{\xi\xi}\right)\right]}\right\}$$
$$-\frac{1}{27}\left\{\frac{\left[K_1^3\gamma_\xi^4 + \left(f+\frac{\lambda_\xi}{\lambda}\right)^{-1} K_1\gamma_\xi\left(K_{1\xi}\gamma_\xi + K_1\gamma_{\xi\xi}\right)\right]}{\left[K_1^2\gamma_\xi^2\left(K_{1\xi}\gamma_\xi + K_1\gamma_{\xi\xi}\right)\right]}\right\}^3 \qquad (12f)$$

With the relations $s_1=\left[N+\left(M^3+N^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}$ and $s_2=\left[N-\left(M^3+N^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}$, the root of (12d) is written as,

$$\gamma=\int_\xi \lambda e^{\int_\xi fd\xi}d\xi=\left(s_1+s_2\right)-\frac{1}{3}\frac{\left[K_1^3\gamma_\xi^4+\left(f+\frac{\lambda_\xi}{\lambda}\right)^{-1}K_1\gamma_\xi\left(K_{1\xi}\gamma_\xi+K_1\gamma_{\xi\xi}\right)\right]}{\left[K_1^2\gamma_\xi^2\left(K_{1\xi}\gamma_\xi+K_1\gamma_{\xi\xi}\right)\right]} \tag{12g}$$

This will solve the integral in (9a). Therefore, the following theorem is just proved,

Theorem 3*: Consider the following integral equation,*

$$B=\int_\xi \lambda(\xi)e^{\int_\xi f(\xi)d\xi}d\xi$$

There exists a functional γ *and* χ *which are defined by,*

$$\gamma=\int_\xi \lambda e^{\int_\xi fd\xi}d\xi \ \ and \ \ \chi=-\frac{1}{(n-1)\lambda^n e^{n\int_\xi fd\xi}}\frac{\gamma_\xi}{\gamma}$$

such that the integral B *can be evaluated as,*

$$\int_\xi \lambda e^{\int_\xi fd\xi}d\xi=\left[N+\left(M^3+N^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}+\left[N-\left(M^3+N^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}-\frac{1}{3}\frac{\left[K_1^3\gamma_\xi^4+\left(f+\frac{\lambda_\xi}{\lambda}\right)^{-1}K_1\gamma_\xi\left(K_{1\xi}\gamma_\xi+K_1\gamma_{\xi\xi}\right)\right]}{\left[K_1^2\gamma_\xi^2\left(K_{1\xi}\gamma_\xi+K_1\gamma_{\xi\xi}\right)\right]}$$

where $K_1=-\frac{1}{\lambda^2 e^{2\int_\xi fd\xi}}$, M *and* N *are defined by (12e) and (12f).*

Examples;

Now, the examples of the proposed integral evaluation taken from the integral table are given [13]. Consider the integral,

$$y=\int_x x^2e^{ax}dx=\frac{a^2x^2-2ax+2}{a^3}e^{ax}$$

where according to theorem 3 the functions λ and f are x^2 and a respectively. The comparison are shown as in the following,

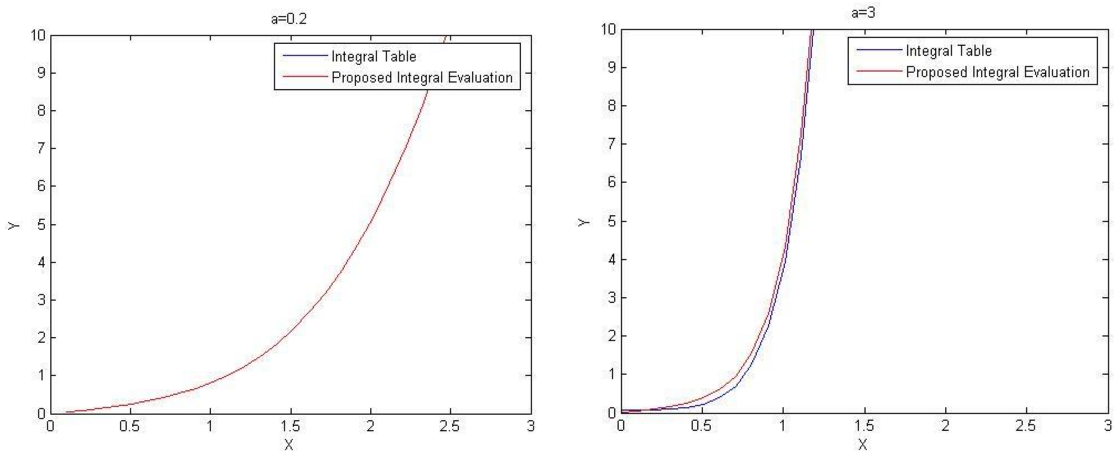

Figure 1. The comparion of the known integral formula againts the proposed integral evaluation

Figure 1 shows that the computations of the proposed integral evaluation for the spesific integral equation are very close to the known result, even coincide for certain constant coefficient.

5. Conclusions

The method of reduction of the higher order linear ordinary differential equations is proposed in this article. The main strategy is to decompose the coefficients and the process thus continued until second order equation is obtained and solved. The procedure for solving second order equation and exact integral evaluation are also conducted and developed to complete the solutions. The paper have ilustrated the new idea of coefficient decompositition to solve the general ODEs with variable coefficients. It is shown that the method can obtain the solutions of arbitrary coefficients and arbitrary order higher than one in closed-form. Moreover, the new formulation of integral evaluation will make the results are tractable for computer simulations. We plan to conduct the applications in our future works.

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