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More on some inequalities for the digamma function

Banyat Sroysang

Department of Mathematics and Statistics,
Faculty of Science and Technology,
Thammasat University, Pathumthani 12121 Thailand
banyat@mathstat.sci.tu.ac.th

Abstract

In this paper, we generalize some inequalities for the digamma function.

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1 Introduction

The digamma function Ψ is defined by

$$\Psi(x) = \frac{d}{dx} \ln \Gamma(x),$$

where Γ is the gamma function and $x > 0$.

Abramowitz and Stegun [1] proved that, for all $x > 0$,

$$\Psi(x) = -\gamma + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{x+k} \right),$$

where γ is the Euler constant. Thus, for all $x > 0$,

$$\Psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}}.$$

where n is a positive integer.

In 2011, Sulaiman [3] presented three inequalities as follows.

$$\Psi(x+y) \geq \Psi(x) + \Psi(y) \tag{1}$$

where $x > 0$ and $0 < y < 1$.

$$\Psi^{(n)}(x+y) \leq \Psi^{(n)}(x) + \Psi^{(n)}(y) \tag{2}$$

where n is a positive odd integer and $x, y > 0$.

$$\Psi^{(n)}(x+y) \geq \Psi^{(n)}(x) + \Psi^{(n)}(y) \tag{3}$$

where n is a positive even integer and $x, y > 0$.

In 2013, Sroysang [2] generalized the inequalities (1), (2) and (3) as follows.

$$\Psi(x + \sum_{i=1}^{m} y_i) \geq \Psi(x) + \sum_{i=1}^{m} \Psi(y_i). \tag{4}$$

where $x > 0$ and $0 < y_i \leq 1$ for all $i \in \mathbb{N}_m$.

$$\Psi^{(n)}\left(x + \sum_{i=1}^{m} y_i\right) \leq \Psi^{(n)}(x) + \sum_{i=1}^{m} \Psi^{(n)}(y_i). \tag{5}$$

where n be a positive odd integer and $x > 0$, $y_i > 0$ for all $i \in \mathbb{N}_m$.

$$\Psi^{(n)}\left(x + \sum_{i=1}^{m} y_i\right) \geq \Psi^{(n)}(x) + \sum_{i=1}^{m} \Psi^{(n)}(y_i). \tag{6}$$

where n be a positive even integer and $x > 0$, $y_i > 0$ for all $i \in \mathbb{N}_m$.

In this paper, we present the generalizations for the inequalities (4), (5) and (6).

2 Results

Theorem 2.1. *Assume that $x > 0$, $\beta_i > 0$ and $0 < y_i \leq 1$ for all $i \in \mathbb{N}_m$. Then*

$$\Psi(x + \sum_{i=1}^{m} \beta_i y_i) \geq \Psi(x) + \sum_{i=1}^{m} \beta_i \Psi(y_i). \tag{7}$$

Proof. Let $f(x) = \Psi(x + \sum_{i=1}^{m} \beta_i y_i) - \Psi(x) - \sum_{i=1}^{m} \beta_i \Psi(y_i)$. Then

$$\begin{aligned} f'(x) &= \Psi'(x+\sum_{i=1}^{m}\beta_i y_i) - \Psi'(x) \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{(x+\sum_{i=1}^{m}\beta_i y_i + k)^2} - \frac{1}{(x+k)^2}\right) \le 0. \end{aligned}$$

Thus, f is non-increasing. Moreover,

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \gamma\sum_{i=1}^{m}\beta_i \\ &\quad + \lim_{x\to\infty}\sum_{k=0}^{\infty}\left(\sum_{i=1}^{m}\frac{-\beta_i}{k+1} - \frac{1}{x+\sum_{i=1}^{m}\beta_i y_i + k} + \frac{1}{x+k} + \sum_{i=1}^{m}\frac{\beta_i}{y_i+k}\right) \\ &= \gamma\sum_{i=1}^{m}\beta_i + \sum_{k=0}^{\infty}\left(\sum_{i=1}^{m}\frac{-\beta_i}{k+1} + \sum_{i=1}^{m}\frac{\beta_i}{y_i+k}\right) \\ &= \gamma\sum_{i=1}^{m}\beta_i + \sum_{k=0}^{\infty}\sum_{i=1}^{m}\frac{\beta_i\,(1-y_i)}{(k+1)(y_i+k)} \ge 0. \end{aligned}$$

It follows that $f(x) \ge 0$ and then we obtain the inequality (7). ☐

Theorem 2.2. *Let n be a positive integer. Assume that $x > 0$, $\beta_i > 0$ and $y_i > 0$ for all $i \in \mathbb{N}_m$. It follows that*
(i) if n is odd, then

$$\Psi^{(n)}\left(x+\sum_{i=1}^{m}\beta_i y_i\right) \le \Psi^{(n)}(x) + \sum_{i=1}^{m}\beta_i\Psi^{(n)}(y_i), \tag{8}$$

and (ii) if n is even, then

$$\Psi^{(n)}\left(x+\sum_{i=1}^{m}\beta_i y_i\right) \ge \Psi^{(n)}(x) + \sum_{i=1}^{m}\beta_i\Psi^{(n)}(y_i). \tag{9}$$

Proof. Let $f(x) = \Psi^{(n)}(x) + \sum_{i=1}^{m}\beta_i\Psi^{(n)}(y_i) - \Psi^{(n)}\left(x+\sum_{i=1}^{m}\beta_i y_i\right)$. Then

$$f'(x) = \Psi^{(n+1)}(x) - \Psi^{(n+1)}\left(x + \sum_{i=1}^{m}\beta_i y_i\right)$$
$$= (n+1)!\sum_{k=0}^{\infty}\left(-\frac{1}{(x+k)^{n+2}} + \frac{1}{(x+\sum_{i=1}^{m}\beta_i y_i + k)^{n+2}}\right).$$

If n is odd, then $f'(x) \leq 0$ and then f is non-increasing. If n is even, then $f'(x) \geq 0$ and then f is non-decreasing.

Moreover,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} n!\times$$
$$\sum_{k=0}^{\infty}\left(\frac{1}{(x+k)^{n+1}} + \sum_{i=1}^{m}\frac{\beta_i}{(y_i+k)^{n+1}} - \frac{1}{(x+\sum_{i=1}^{m}\beta_i y_i + k)^{n+1}}\right)$$
$$= n!\sum_{k=0}^{\infty}\sum_{i=1}^{m}\frac{\beta_i}{(y_i+k)^{n+1}}.$$

If n is odd, then $\lim_{x\to\infty} f(x) \geq 0$, so $f(x) \geq 0$ and then we obtain the inequality (8). If n is even, then $\lim_{x\to\infty} f(x) \leq 0$ and then $f(x) \leq 0$, so we obtain the inequality (9). □

References

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