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Inequalities for the k-th derivative of the incomplete exponential integral function

Banyat Sroysang

Department of Mathematics and Statistics,
Faculty of Science and Technology,
Thammasat University, Pathumthani 12121 Thailand
banyat@mathstat.sci.tu.ac.th

Abstract

In this paper, we present some inequalities for the n-th derivative of the incomplete exponential integral function.

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1 Introduction

The exponential integral function [1, 3] is defined by

$$E_n(x) = \int_1^\infty t^{-n} e^{-xt} dt$$

where $x > 0$ and $n \in \mathbb{N}_0$.

For any $k \in \mathbb{N}$, the k-derivative of the exponential integral function E_n is given by

$$E_n^{(k)}(x) = (-1)^k \int_1^\infty t^{k-n} e^{-xt} dt$$

where $x > 0$ and $n \in \mathbb{N}_0$.

The incomplete exponential integral function is defined by

$${}_a^bE_n(x) = \int_a^b t^{-n} e^{-xt} dt$$

where $x > 0$, $1 < a < b$ and $n \in \mathbb{N}_0$.

For any $k \in \mathbb{N}$, the k-derivative of the incomplete exponential integral function E_n is given by

$$ {}_a^bE_n^{(k)}(x) = (-1)^k \int_a^b t^{k-n} e^{-xt} dt $$

where $x > 0$ and $n \in \mathbb{N}_0$.

In 2012, Sulaiman [3] gave the inequalities involving the n-th derivative of the exponential integral functions as follows.

For any $x, y > 0$, $p > 1 = \frac{1}{p} + \frac{1}{q}$, $m + n, pm, qn \in \mathbb{N}_0$, and k is an even integer such that $k > m + n$,

$$ E_{m+n}^{(k)} \left(\frac{x}{p} + \frac{y}{q} \right) \leq \left(E_{pm}^{(k)}(x) \right)^{1/p} \left(E_{qn}^{(k)}(y) \right)^{1/q}. \tag{1} $$

For any $x > 0$, $0 < y \leq 1$, $n \in \mathbb{N}_0$, $p > 1$, $0 < r < 1$ and $\frac{1}{p} + \frac{1}{q} = 1 = \frac{1}{r} + \frac{1}{s}$, and k is an even integer such that $k > n$,

$$ E_n^{(k)}(xy) \geq \left(E_n^{(k)} \left(\frac{rx^p}{p} \right) \right)^{1/r} \left(E_n^{(k)} \left(\frac{sy^q}{q} \right) \right)^{1/s}. \tag{2} $$

In 2013, Sroysang [2] presented the generalizations for the inequalities (1) and (2).

In this paper, we present two inequalities for the k-derivative of the incomplete exponential integral function similar to the inequalities (1) and (2).

2 Results

Theorem 2.1. *Assume that* $1 < a < b$, $x > 0$, $y > 0$, $p > 1 = \frac{1}{p} + \frac{1}{q}$, $\{m + n, pm, qn\} \subseteq \mathbb{N}_0$, *and* k *is an even integer such that* $k > m + n$. *Then*

$$ {}_a^bE_{m+n}^{(k)} \left(\frac{x}{p} + \frac{y}{q} \right) \leq \left({}_a^bE_{pm}^{(k)}(x) \right)^{1/p} \left({}_a^bE_{qn}^{(k)}(y) \right)^{1/q}. $$

Proof. By the Hölder inequality,

$$
\begin{aligned}
{}_a^bE_{m+n}^{(k)}\left(\frac{x}{p}+\frac{y}{q}\right) &= (-1)^k \int_a^b t^{k-(m+n)} e^{-t\left(\frac{x}{p}+\frac{y}{q}\right)} dt \\
&= \int_a^b t^{k-(m+n)} e^{-t\left(\frac{x}{p}+\frac{y}{q}\right)} dt \\
&= \int_a^b t^{k\left(\frac{1}{p}+\frac{1}{q}\right)-(m+n)} e^{-t\left(\frac{x}{p}+\frac{y}{q}\right)} dt \\
&= \int_a^b \left(t^{\frac{k}{p}-m} e^{-t\frac{x}{p}t}\right)\left(t^{\frac{k}{q}-n} e^{-t\frac{y}{q}t}\right) dt \\
&\le \left(\int_a^b t^{k-pm} e^{-xt} dt\right)^{1/p} \left(\int_a^b t^{k-qn} e^{-yt} dt\right)^{1/q} \\
&= \left((-1)^k \int_a^b t^{k-pm} e^{-xt} dt\right)^{1/p} \left((-1)^k \int_a^b t^{k-qn} e^{-yt} dt\right)^{1/q} \\
&= \left({}_a^bE_{pm}^{(k)}(x)\right)^{1/p} \left({}_a^bE_{qn}^{(k)}(y)\right)^{1/q}.
\end{aligned}
$$

□

Corollary 2.2. *Assume that $1 < a < b$, $x > 0$, $y > 0$, $\{m, n\} \subseteq \mathbb{N}_0$, and k is an even integer such that $k > m + n$. Then*

$$
\left[{}_a^bE_{m+n}^{(k)}\left(\frac{x+y}{2}\right)\right]^2 \le \left({}_a^bE_{2m}^{(k)}(x)\right)\left({}_a^bE_{2n}^{(k)}(y)\right).
$$

Theorem 2.3. *Assume that $0 < x, y < 1 < a < b$ and $0 < r, s < 1 < p$ and $n \in \mathbb{N}_0$, $\frac{1}{p}+\frac{1}{q} = 1 = \frac{1}{r}+\frac{1}{s}$ and k is an even integer such that $k > n$. Then*

$$
{}_a^bE_n^{(k)}(xy) \ge \left({}_a^bE_n^{(k)}\left(\frac{rx^p}{p}\right)\right)^{1/r} \left({}_a^bE_n^{(k)}\left(\frac{sy^q}{q}\right)\right)^{1/s}.
$$

Proof. For any $z > 0$,

$$
{}_a^bE_n^{(k+1)}(z) = (-1)^{k+1} \int_a^b t^{k+1-n} e^{-zt} dt = -\int_a^b t^{k+1-n} e^{-zt} dt \le 0.
$$

This implies that ${}_a^bE_m^{(k)}$ is non-increasing.
Note that

$$
xy \le \frac{x^p}{p} + \frac{y^q}{q}.
$$

By the reverse Hölder inequality,

$$
\begin{aligned}
{}_a^bE_n^{(k)}(xy) &\geq_a^b E_n^{(k)}\left(\frac{x^p}{p}+\frac{y^q}{q}\right) \\
&= (-1)^k \int_a^b t^{k-n} e^{-t\left(\frac{x^p}{p}+\frac{y^q}{q}\right)} dt \\
&= \int_a^b t^{k-n} e^{-t\left(\frac{x^p}{p}+\frac{y^q}{q}\right)} dt \\
&= \int_a^b \left(t^{\frac{k-n}{r}} e^{-\frac{x^p}{p}t}\right)\left(t^{\frac{k-n}{s}} e^{-\frac{y^q}{q}t}\right) dt \\
&\geq \left(\int_a^b t^{k-n} e^{\frac{-rx^p}{p}t} dt\right)^{1/r} \left(\int_a^b t^{k-n} e^{\frac{-sy^q}{q}t} dt\right)^{1/s} \\
&=\geq \left((-1)^k \int_a^b t^{k-n} e^{\frac{-rx^p}{p}t} dt\right)^{1/r} \left((-1)^k \int_a^b t^{k-n} e^{\frac{-sy^q}{q}t} dt\right)^{1/s} \\
&= \left({}_a^bE_n^{(k)}\left(\frac{rx^p}{p}\right)\right)^{1/r} \left({}_a^bE_n^{(k)}\left(\frac{sy^q}{q}\right)\right)^{1/s}.
\end{aligned}
$$

□

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