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More on some inequalities for the incomplete exponential integral function

Banyat Sroysang

Department of Mathematics and Statistics,
Faculty of Science and Technology,
Thammasat University, Pathumthani 12121 Thailand
banyat@mathstat.sci.tu.ac.th

Abstract

In this paper, we generalize some inequalities involving the incomplete exponential integral function.

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1 Introduction

The exponential integral function [1, 2] is defined by

$$E_n(x) = \int_1^\infty t^{-n} e^{-xt} dt$$

where $x > 0$ and $n \in \mathbb{N}_0$.

The incomplete exponential integral function is defined by

$${}_a^bE_n(x) = \int_a^b t^{-n} e^{-xt} dt$$

where $x > 0$, $1 < a < b$ and $n \in \mathbb{N}_0$.

In 2013, Sroysang [3] proved that the incomplete exponential integral function is non-increasing and then gave the inequalities as follows..

$${}_a^bE_{m+n}\left(\frac{x}{p} + \frac{y}{q}\right) \leq \left({}_a^bE_{pm}(x)\right)^{1/p} \left({}_a^bE_{qn}(y)\right)^{1/q} \tag{1}$$

where $1 < a < b$, $x, y > 0$, $p > 1 = \frac{1}{p} + \frac{1}{q}$, and $m + n, pm, qn \in \mathbb{N}_0$.

$$ {}_a^bE_n(xy) \leq \left({}_a^bE_n(px)\right)^{1/p} \left({}_a^bE_n(qy)\right)^{1/q} \tag{2} $$

where $1 < a < b$, $x, y > 1$, $n \in \mathbb{N}_0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $x + y \leq xy$.

$$ {}_a^bE_n(xy) \geq \left({}_a^bE_n(px)\right)^{1/p} \left({}_a^bE_n(qy)\right)^{1/q} \tag{3} $$

where $1 < a < b$, $x > 0$, $0 < y < 1$, $n \in \mathbb{N}_0$, $0 < p < 1 = \frac{1}{p} + \frac{1}{q}$ and $x + y \geq xy$.

$$ {}_a^bE_n(xy) \geq \left({}_a^bE_n\left(\frac{rx^p}{p}\right)\right)^{1/r} \left({}_a^bE_n\left(\frac{sy^q}{q}\right)\right)^{1/s} \tag{4} $$

$1 < a < b$, $x, y > 1$, $n \in \mathbb{N}_0$, $p > 1$, $0 < r < 1$ and $\frac{1}{p} + \frac{1}{q} = 1 = \frac{1}{r} + \frac{1}{s}$.

In this paper, we generalize the inequalities (1), (2), (3) and (4).

2 Results

Theorem 2.1. *Let $x_i > 0$, $n_i \geq 0$ and $p_i > 1$ be such that $p_i n_i \in \mathbb{N}_0$ for all $i \in \mathbb{N}_m$. Assume that $1 < a < b$, $\sum_{i=1}^{m} p_i = 1$ and $\sum_{i=1}^{m} n_i \in \mathbb{N}_0$. Then*

$$ {}_a^bE_{\sum_{i=1}^m n_i}\left(\sum_{i=1}^{m} \frac{x_i}{p_i}\right) \leq \prod_{i=1}^{m} \left({}_a^bE_{p_i n_i}(x)\right)^{1/p_i}. $$

Proof. By the generalized Hölder inequality,

$$ \begin{aligned} {}_a^bE_{\sum_{i=1}^m n_i}\left(\sum_{i=1}^{m} \frac{x_i}{p_i}\right) &= \int_a^b t^{-\sum_{i=1}^m n_i} e^{-t\sum_{i=1}^m \frac{x_i}{p_i}} dt \\ &= \int_a^b \left(\prod_{i=1}^{m} t^{-n_i} e^{-t\frac{x_i}{p_i}}\right) dt \\ &\leq \prod_{i=1}^{m} \left(\int_a^b t^{-p_i n_i} e^{-x_i t} dt\right)^{1/p_i} \\ &= \prod_{i=1}^{m} \left({}_a^bE_{p_i n_i}(x)\right)^{1/p_i}. \end{aligned} $$

□

Theorem 2.2. *Let $n \in \mathbb{N}_0$, $x_i > 1$ and $p_i > 1$ for all $i \in \mathbb{N}_m$. Assume that $1 < a < b$, $\sum_{i=1}^{m} p_i = 1$ and $\sum_{i=1}^{m} x_i \leq \prod_{i=1}^{m} x_i$. Then*

$$ {}_a^bE_n\left(\prod_{i=1}^{m} x_i\right) \leq \prod_{i=1}^{m} \left({}_a^bE_n(p_i x_i)\right)^{1/p_i}. $$

Proof. In [3], we obtain that ${}_a^bE_n$ is non-increasing. By the generalized Hölder inequality,

$$\begin{aligned}
{}_a^bE_n\left(\prod_{i=1}^m x_i\right) &\leq {}_a^b E_n\left(\sum_{i=1}^m x_i\right) \\
&= \int_a^b t^{-n} e^{-t\sum_{i=1}^m x_i} dt \\
&= \int_a^b \left(\prod_{i=1}^m t^{-\frac{n}{p_i}} e^{-x_i t}\right) dt \\
&\leq \prod_{i=1}^m \left(\int_a^b t^{-n} e^{-p_i x_i t} dt\right)^{1/p_i} \\
&= \prod_{i=1}^m \left({}_a^bE_n(p_i x_i)\right)^{1/p_i}.
\end{aligned}$$

□

Theorem 2.3. *Let* $n \in \mathbb{N}_0$, $0 < x_i < 1$ *and* $0 < p_i < 1$ *for all* $i \in \mathbb{N}_m$. *Assume that* $1 < a < b$, $\sum_{i=1}^m p_i = 1$ *and* $\sum_{i=1}^m x_i \geq \prod_{i=1}^m x_i$. *Then*

$${}_a^bE_n\left(\prod_{i=1}^m x_i\right) \leq \prod_{i=1}^m \left({}_a^bE_n(p_i x_i)\right)^{1/p_i}.$$

Proof. In [3], we obtain that ${}_a^bE_n$ is non-increasing. By the generalized reverse Hölder inequality,

$$\begin{aligned}
{}_a^bE_n\left(\prod_{i=1}^m x_i\right) &\geq {}_a^b E_n\left(\sum_{i=1}^m x_i\right) \\
&= \int_a^b t^{-n} e^{-t\sum_{i=1}^m x_i} dt \\
&= \int_a^b \left(\prod_{i=1}^m t^{-\frac{n}{p_i}} e^{-x_i t}\right) dt \\
&\geq \prod_{i=1}^m \left(\int_a^b t^{-n} e^{-p_i x_i t} dt\right)^{1/p_i} \\
&= \prod_{i=1}^m \left({}_a^bE_n(p_i x_i)\right)^{1/p_i}.
\end{aligned}$$

□

Theorem 2.4. *Let $n \in \mathbb{N}_0$, $x_i > 1$, $p_i > 1$, and $0 < r_i < 1$ for all $i \in \mathbb{N}_m$. Assume that $1 < a < b$ and $\sum_{i=1}^{m} p_i = 1 = \sum_{i=1}^{m} r_i$. Then*

$$ {}_a^bE_n\left(\prod_{i=1}^{m} x_i\right) \geq \prod_{i=1}^{m}\left({}_a^bE_n\left(\frac{r_i x_i^{p_i}}{p_i}\right)\right)^{1/r_i}. $$

Proof. We note that

$$ \prod_{i=1}^{m} x_i \leq \sum_{i=1}^{m} \frac{x_i^{p_i}}{p_i}. $$

In [3], we obtain that ${}_a^bE_n$ is non-increasing. By the generalized reverse Hölder inequality,

$$ \begin{aligned} {}_a^bE_n\left(\prod_{i=1}^{m} x_i\right) &\geq {}_a^b E_n\left(\sum_{i=1}^{m} \frac{x_i^{p_i}}{p_i}\right) \\ &= \int_a^b t^{-n} e^{-t\sum_{i=1}^{m}\frac{x_i^{p_i}}{p_i}}\,dt \\ &= \int_a^b \left(\prod_{i=1}^{m} t^{-\frac{n}{r_i}} e^{-\frac{x_i^{p_i}}{p_i}t}\right) dt \\ &\geq \prod_{i=1}^{m}\left(\int_a^b t^{-n} e^{-\frac{r_i x_i^{p_i}}{p_i}t}\,dt\right)^{1/r_i} \\ &= \prod_{i=1}^{m}\left({}_a^bE_n\left(\frac{r_i x_i^{p_i}}{p_i}\right)\right)^{1/r_i}. \end{aligned} $$

□

References

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