Mathematica Aeterna, Vol. 4, 2014, no. 2, 135 - 141

Local Regularity for Solutions to Divergence Type Elliptic Equations with Advection and Lower-order Terms

Gao Hongya

College of Mathematics and Computer Science, Hebei University,
Baoding, 071002, China. Email: ghy@hbu.cn

Wang Tian

College of Mathematics and Computer Science, Hebei University,
Baoding, 071002, China. Email: 260907818@qq.com

Teng Hongxu

College of Mathematics and Computer Science, Hebei University,
Baoding, 071002, China. Email: 1724769102@qq.com

Abstract

We obtain a local regularity result for distributional solutions to elliptic equations of divergence type with advection and lower-order terms that satisfy appropriate growth conditions.

Mathematics Subject Classification: 35J60.

Keywords: Local regularity, elliptic equation, advection term, lower-order term.

1 Introduction and Main Result.

Let Ω be a bounded open subset of R^n. Let us consider elliptic equations of the form

$$-\mathrm{div}\mathcal{A}(x,u,Du(x)) - \mathrm{div}g(x,u) = h(x,u) - \mathrm{div}F(x) + f(x), \tag{1.1}$$

where the vector field $\mathcal{A} : \Omega \times \mathrm{R} \times \mathrm{R}^n \to \mathrm{R}^n$ is a Carathéodory function satisfying the following structure conditions: for a.e. $x \in \Omega$, all $s \in \mathrm{R}$ and all $\xi \in \mathrm{R}^n$,

$$A(x,s,\xi)\cdot\xi \geq C_{A,1}|\xi|^p, \tag{1.2}$$

$$|\mathcal{A}(x,s,\xi)| \le C_{A,2}|\xi|^{p-1} + C_{A,3}|s|^{p-1} + k_1(x), \tag{1.3}$$

where $0 \le k_1(x) \in L^{r_1}_{loc}(\Omega)$.

The advection field $g : \Omega \times \mathrm{R} \to \mathrm{R}^n$ is a Carathéodory function, and for a.e. $x \in \Omega$ and for all $s \in \mathrm{R}$,

$$|g(x,s)| \le k_2(x) + C_g|s|^{p-1}, \tag{1.4}$$

where $0 \le k_2(x) \in L^{r_2}_{loc}(\Omega)$.

The Carathéodory function $h : \Omega \times \mathrm{R} \to \mathrm{R}$ satisfies

$$|h(x,s)| \le k_3(x) + C_h|s|^{p-1}, \tag{1.5}$$

for a.e. $x \in \Omega$ and all $s \in \mathrm{R}$, where $0 \le k_3(x) \in L^{r_3}_{loc}(\Omega)$. Finally, we assume $F(x) \in L^{r_4}_{loc}(\Omega, \mathrm{R}^n)$ and $f(x) \in L^{r_5}_{loc}(\Omega)$.

We look for distributional solutions to (1.1) in the following sense:

Definition 1.1 *A distributional solution of* (1.1) *is a function* $u \in W^{1,p}_{loc}(\Omega)$ *satisfying*

$$\int_\Omega (\mathcal{A}(x,u,Du) + g(x,u))D\varphi dx = \int_\Omega F(x)D\varphi dx + \int_\Omega (h(x,u) + f(x))\varphi dx, \tag{1.6}$$

for all $\varphi \in W^{1,p}(\Omega)$ *with compact support.*

In [1], Giachetti and Porzio considered distributional solutions $u \in W^{1,p}_{loc}(\Omega)$ to elliptic equation of the form

$$-\mathrm{div}\mathcal{A}(x,u,Du) = -\mathrm{div}F, \tag{1.1$'$}$$

with Carachéodory function $\mathcal{A} : \Omega \times \mathrm{R} \times \mathrm{R}^n \to \mathrm{R}^n$ satisfying the coercivity and growth conditions (1.2) and (1.3), and obtained a local regularity result, see [1, Theorem 5.1]. Some generalizations of the above result can be found in [2-7]. Integrability property is important among the regularity theories of nonlinear elliptic PDEs and systems, see [8-13]. In the present paper, we consider distributional solutions to elliptic equations of type (1.1). The main result of this paper is the following theorem.

Theorem 1.2 *Let* $1 < p < n$. *Under the previous assumptions, if the exponents* r_1, r_2, r_3, r_4 *and* r_5 *satisfy*

$$\frac{p}{p-1} < \min\{r_1, r_2, r_3, r_4, r_5\} < \frac{n}{p-1},$$

then u *belongs to* $L^s_{loc}(\Omega)$, *where* $s = [(p-1)\min\{r_1, r_2, r_3, r_4, r_5\}]^*$.

Notice we have restricted ourselves to the case $p < n$ because when $g \geq n$, every function in $W^{1,p}_{loc}(\Omega)$ is trivially in $L^s_{loc}(\Omega)$ by the Sobolev Theorem.

For $x_0 \in \Omega$ and $t \in \mathrm{R}^+$, we denote by $B_t = B_t(x_0)$ the ball of radius t centered at x_0. For $k > 0$ and a measurable function $u(x)$, we let

$$A_k = \{x \in \Omega : |u(x)| > k\} \text{ and } A_{k,t} = A_k \cap B_t.$$

In order to prove Theorem 1.2, we need two lemmas. The first lemma can be found in [1].

Lemma 1.3 *Let $u \in W^{1,p}_{loc}(\Omega)$, $\varphi_0 \in L^r_{loc}(\Omega)$, where $1 < p < n$ and r satisfies $1 < r < \frac{n}{p}$. Assume that the following integral estimate holds:*

$$\int_{A_{k,\sigma}} |Du|^p dx \leq c_0 \left[\int_{A_{k,\gamma}} \varphi_0 dx + (t-\tau)^{-\alpha} \int_{A_{k,\gamma}} |u|^p dx \right],$$

for every $k \in N$ and $R_0 \leq \sigma < \gamma \leq R_1$, where c_0 is a positive constant that depends only on N, p, r, R_0, R_1 and $|\Omega|$, and α is a real positive constant. Then $u \in L^s_{loc}(\Omega)$, where

$$s = (pr)^*.$$

The following lemma comes from [9].

Lemma 1.4 *Let $f(\tau)$ be a nonnegative bounded function defined for $0 \leq R_0 \leq t \leq R_1$. Suppose that for $R_0 \leq \tau < t \leq R_1$ we have*

$$f(\tau) \leq A(t-\tau)^{-\alpha} + B + \theta f(t),$$

where A, B, α, θ are non-negative constants, and $\theta < 1$. Then there exists a constant c, depending only on α and θ such that for every σ, γ, $R_0 \leq \sigma < \gamma \leq R_1$ we have

$$f(\sigma) \leq c[A(\gamma-\sigma)^{-\alpha} + B].$$

2 Proof of Theorem 1.2.

Let $B_{R_1} \subset\subset \Omega$ and $0 \leq R_0 \leq \tau < t \leq R_1$, be arbitrarily fixed. It is no loss of generality to assume $R_1 < 1$. For $u \in W^{1,p}_{loc}(\Omega)$ a distributional solution of (1.1), we choose $\varphi = \eta(u - T_k(u))$, where η is a cut-off function such that

$$\text{supp}\eta \subset B_t,\ 0 \leq \eta \leq 1, \eta = 1 \text{ in } B_\tau, \text{ and } |D\eta| \leq 2(t-\tau)^{-1},$$

and $T_k(u)$ is the usual truncation of u at level $k > 0$, that is,

$$T_k(u) = \max\{-k, \min\{k, u\}\}.$$

Since

$$D\varphi = D(\eta(u - T_k(u))) = (u - T_k(u))D\eta + \eta D(u - T_k(u)),$$

and $u - T_k(u) = 0$ for $x \in \{|u(x)| \le k\}$, then (1.6) yields

$$\begin{aligned} & \int_{A_{k,t}} \mathcal{A}(x, Du)\eta Du dx \\ = & -\int_{A_{k,t}} \mathcal{A}(x, Du)(u - T_k(u))D\eta dx - \int_{A_{k,t}} g(x, u)\left((u - T_k(u))D\eta + \eta Du\right) dx \\ & + \int_{A_{k,t}} F(x)\left((u - T_k(u))D\eta + \eta Du\right) dx + \int_{A_{k,t}} h(x, u)\eta(u - T_k(u))dx \\ & + \int_{A_{k,t}} f(x)\eta(u - T_k(u))dx \\ = & I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{2.1}$$

Using (1.2), the left-hand side of the above equality can be estimated as

$$\int_{A_{k,t}} \mathcal{A}(x, Du)\eta Du dx \ge C_{A,1} \int_{A_{k,t}} \eta |Du|^p dx \ge C_{A,1} \int_{A_{k,\tau}} |Du|^p dx \tag{2.2}$$

Since $|u - T_k(u)| \le |u|$, then using (1.3), $|I_1|$ can be estimated as

$$\begin{aligned} |I_1| = & \left| -\int_{A_{k,t}} \mathcal{A}(x, Du)(u - T_k(u))D\eta dx \right| \\ \le & 2\int_{A_{k,t}} \left(C_{A,2}|Du|^{p-1} + C_{A,3}|u|^{p-1} + k_1(x) \right) \frac{|u|}{t-\tau} dx \\ \le & 2C_{A,2} \left(\int_{A_{k,t}} |Du|^p dx \right)^{\frac{p-1}{p}} \left(\int_{A_{k,t}} \frac{|u|^p}{(t-\tau)^p} dx \right)^{\frac{1}{p}} \\ & + 2C_{A,3} \int_{A_{k,t}} \frac{|u|^p}{t-\tau} dx + 2\left(\int_{A_{k,t}} k_1(x)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_{A_{k,t}} \frac{|u|^p}{(t-\tau)^p} dx \right)^{\frac{1}{p}} \\ \le & C_{A,2}\varepsilon \int_{A_{k,t}} |Du|^p dx + C(\varepsilon)C_{A,2} \int_{A_{k,t}} \frac{|u|^p}{(t-\tau)^p} dx \\ & + 2C_{A,3} \int_{A_{k,t}} \frac{|u|^p}{(t-\tau)^p} dx + C(\varepsilon) \int_{A_{k,t}} k_1(x)^{\frac{p}{p-1}} dx + \varepsilon \int_{A_{k,t}} \frac{|u|^p}{(t-\tau)^p} dx \\ = & C_{A,2}\varepsilon \int_{A_{k,t}} |Du|^p dx + (C(\varepsilon)C_{A,2} + 2C_{A,3} + \varepsilon) \int_{A_{k,t}} \frac{|u|^p}{(t-\tau)^p} dx \\ & + C(\varepsilon) \int_{A_{k,t}} k_1(x)^{\frac{p}{p-1}} dx, \end{aligned} \tag{2.3}$$

where we have used Hölder inequality, Young inequality and the fact $t < R_1 < 1$, which implies $1 < \frac{1}{t-\tau} < \frac{1}{(t-\tau)^p}$.

Using (1.4), Hölder inequality and Young inequality, $|I_2|$ can be estimated as

$$\begin{aligned}
|I_2| &= \left|-\int_{A_{k,t}} g(x,u)\left((u-T_k(u))D\eta+\eta Du\right)dx\right| \\
&\le \int_{A_{k,t}} \left(k_2(x)+C_g|u|^{p-1}\right)\left(\frac{2|u|}{t-\tau}+|Du|\right)dx \\
&\le \left(\int_{A_{k,t}} k_2(x)^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_{A_{k,t}}\left(\frac{2|u|}{t-\tau}+|Du|\right)^p dx\right)^{\frac{1}{p}} \\
&\quad +2C_g\int_{A_{k,t}}\frac{|u|^p}{t-\tau}dx+C_g\left(\int_{A_{k,t}}|u|^p dx\right)^{\frac{p-1}{p}}\left(\int_{A_{k,t}}|Du|^p dx\right)^{\frac{1}{p}} \\
&\le (2^p+C_g)\varepsilon\int_{A_{k,t}}|Du|^p dx+(2^p\varepsilon+C_g(C(\varepsilon)+2))\int_{A_{k,t}}\frac{|u|^p}{(t-\tau)^p}dx \\
&\quad +C(\varepsilon)\int_{A_{k,t}}k_2(x)^{\frac{p}{p-1}}dx.
\end{aligned} \tag{2.4}$$

$|I_3|$ can be estimated as

$$\begin{aligned}
|I_3| &= \left|-\int_{A_{k,t}} F(x)\left((u-T_k(u))D\eta+\eta Du\right)dx\right| \\
&\le C(\varepsilon)\int_{A_{k,t}}|F(x)|^{\frac{p}{p-1}}dx+\varepsilon\int_{A_{k,t}}\left(\frac{2|u|}{t-\tau}+|Du|\right)^p dx \\
&\le C(\varepsilon)\int_{A_{k,t}}|F(x)|^{\frac{p}{p-1}}dx+2^p\varepsilon\int_{A_{k,t}}\frac{|u|^p}{(t-\tau)^p}dx+2^p\varepsilon\int_{A_{k,t}}|Du|^p dx.
\end{aligned} \tag{2.5}$$

Using (1.5), Hölder inequality and Young inequality, $|I_4|$ can be estimated as

$$\begin{aligned}
|I_4| &= \left|\int_{A_{k,t}} h(x,u)\eta(u-T_k(u))dx\right| \\
&\le \int_{A_{k,t}}(k_3(x)+C_h|u|^{p-1})|u|dx \\
&\le C(\varepsilon)\int_{A_{k,t}}k_3(x)^{\frac{p}{p-1}}dx+(\varepsilon+C_h)\int_{A_{k,t}}\frac{|u|^p}{(t-\tau)^p}dx.
\end{aligned} \tag{2.6}$$

$|I_5|$ can be estimated as

$$\begin{aligned}
|I_5| &= \left|\int_{A_{k,t}}|f(x)|\eta(u-T_k(u))dx\right| \le \int_{A_{k,t}}|f(x)||u|dx \\
&\le C(\varepsilon)\int_{A_{k,t}}|f(x)|^{\frac{p}{p-1}}dx+\varepsilon\int_{A_{k,t}}\frac{|u|^p}{(t-\tau)^p}dx.
\end{aligned} \tag{2.7}$$

Combining (2.1)-(2.7) we arrive at

$$\begin{aligned} & C_{A,1}\int_{A_{k,\tau}}|Du|^p dx \\ \leq\ & (C_{A,2}+2^{p+1}+C_g)\varepsilon\int_{A_{k,t}}|Du|^p dx \\ & +(C(\varepsilon)C_{A,2}+2C_{A,3}+\varepsilon+2^p\varepsilon+2^p\varepsilon+C_g(C(\varepsilon)+2)+2\varepsilon+C_h)\int_{A_{k,t}}\frac{|u|^p}{(t-\tau)^p}dx \\ & +C(\varepsilon)\int_{A_{k,t}}(k_1(x)+k_2(x)+k_3(x)+|F(x)|+|f(x)|)^{\frac{p}{p-1}}dx \end{aligned} \tag{2.8}$$

Take ε small enough such that $\frac{(C_{A,2}+2^{p+1}+C_g)\varepsilon}{C_{A,1}}<1$, then Lemma 1.4 yields that for every σ,γ, $R_0\leq\sigma<\gamma\leq R_1$, we have

$$\int_{A_{k,\sigma}}|Du|^p dx\leq C\left[\int_{A_{k,\gamma}}\frac{|u|^p}{(\gamma-\sigma)^p}dx+\int_{A_{k,\gamma}}(k_1(x)+k_2(x)+k_3(x)+|F(x)|+|f(x)|)^{\frac{p}{p-1}}dx\right],$$

where C is a constant depends only on $p, C_{A,1}, C_{A,2}, C_{A,3}, c_g$ and c_h. Theorem 1.2 follows from Lemma 1.3.

ACKNOWLEDGEMENTS. Research supported by NSFC (11371050).

References

[1] Giachetti D, Porzio M M. Local regularity results for minima of functionals of the calculus of variation, Nonlinear Analysis, T.M.A, 2000, 39: 463-482.

[2] Gao H Y, Chu Y M, Quasiregular mappings and $\mathcal{A}$-harmonic equations, Science Press, Beijing, 2013.

[3] Gao H Y, Guo J, Zuo Y L, Chu Y M. Local regularity result in obstacle problems, Acta Math. Sci., 2010, 30B(1): 208-214

[4] Gao H Y, Huang Q H. Local regularity for solutions of anisotropic obstacle problems, Nonlinear Analysis, T.M.A., 2012, 75: 4761-4765

[5] Gao H Y, Qiao J J, Chu Y M. Local regularity and local boundedness results for very weak solutions to obstacle problems, J. Ineq. Appl., 2010, Article ID: 878769, 12 pages, doi:10.1155/2010/878769.

[6] Gao H Y, Qiao J J, Wang Y, Chu Y M. Local reguarity results for minima of anisotropic functionals and solutions of anisotropic equations, J. Ineq. Appl., 2008 (2008) Article ID 835736: 1-12

[7] Gao H Y, Tian H Y. Local regularity result for solutions of obstacle problems, Acta Math. Sci., 2004, 24B(1): 71-74

[8] Bensoussan A, Frehse J. Regularity results for nonlinear elliptic systems and applications, Springer, 2002.

[9] Giaquinta M, Multiple integrals in the calculus of variations and nonlinear elliptic systems, Princeton University Press, Princeton, NJ, 1983.

[10] Giaquinta M, Giusti E. On the regularity of the minima of variational integrals, Acta Math., 1982, 148: 31-46

[11] Gilbarg D, Trudinger N S. Elliptic partial differential equations of second order, Springer, 1977.

[12] Ladyženskaya O A, Ural'ceva N N. Linear and quasilinear elliptic equations, Academic Press, 1968.

[13] Morrey C B. Multiple integrals in the calculus of variations, Springer, 1968.

Received: January, 2014