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3-Lie Algebras and Cubic Matrices

BAI Ruipu

College of Mathematics and Computer Science,
Hebei University, Baoding, 071002, China
email: bairuipu@hbu.cn

Li Qiyong

College of Mathematics and Computer Science,
Hebei University, Baoding, 071002, China

Cheng Rong

College of Mathematics and Computer Science,
Hebei University, Baoding, 071002, China

Abstract

The realization of n-Lie algebras is very important in the study of the structure of n-Lie algebras for $n \geq 3$. This paper considers the realizations of 3-Lie algebras by cubic matrices. First, the trace function tr_1 of cubic matrices is defined, and then the 3-ary Lie multiplication $[, ,]_{tr_1}$ on the vector space Ω spanned by cubic matrices is constructed, and the structure of the 3-Lie algebra $(\Omega, [, ,]_{tr_1})$ is investigated. It is proved that $(\Omega, [, ,]_{tr_1})$ is a decomposable 3-Lie algebra, and there does not exist any metric on it.

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1 Introduction

n-Lie algebras [1] are a kind of multiple algebraic systems appearing in many fields in mathematics and mathematical physics (cf [2, 3, 4]). In this paper, we pay our main attention to construct 3-Lie algebras. In papers [1, 5, 6, 7, 8] 3-Lie algebras are constructed by commutative associative algebras with derivations, γ-matrices, 2-dimensional extensions of metric Lie algebras, associative algebras, Lie algebras and linear functions. Authors in paper [7] defined cubic matrices $A = (a_{ijk})$ for $1 \leq i, j, k \leq 2$, and R. Kerner [9] defined the ternary multiplication of three cubic matrices $(A \oslash B \oslash C)_{ijk} = \sum\limits_{pqr} a_{ipq}b_{pjr}c_{rqk}$, $(A * B * C)_{ijk} = \sum\limits_{pqr} a_{piq}b_{qjr}c_{rkj}$, and the symmetry properties of the ternary algebras are studied. The paper [10] defined five non-isomorphic 2-ary multiplication on the vector space Ω spanned by the cubic matrices $A = (a_{ijk})$ for $1 \leq i, j, k \leq N$, and constructed non-isomorphic 3-Lie algebras. In this paper, we define the new trace function tr_1 of cubic matrices, and construct new 3-Lie algebra $(\Omega, [, ,]_{tr_1})$, which is not isomorphic to the 3-Lie algebras in paper [10]. First we introduce some notions.

An *n-Lie algebra* [1] is a vector space V over a field F equipped with an n-multilinear operation $[x_1, \cdots, x_n]$ satisfying

$$[x_1, \cdots, x_n] = sign(\sigma)[x_{\sigma(1)}, \cdots, x_{\sigma(n)}],$$

$$[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^{n}[x_1, \cdots, [x_i, y_2, \cdots, y_n], \cdots, x_n]$$

for $x_1, \cdots, x_n, y_2, \cdots, y_n \in V$ and $\sigma \in S_n$, the permutation group on n letters.

Denote by $[V_1, V_2, \cdots, V_n]$ the subspace of V generated by all vectors $[x_1, x_2, \cdots, x_n]$, where $x_i \in V_i$, for $i = 1, 2, \cdots, n$. The subalgebra $V^1 = [V, V, \cdots, V]$ is called *the derived algebra* of V. If $V^1 = 0$, then V is an abelian n-Lie algebra.

An *ideal* of an n-Lie algebra V is a subspace I such that $[I, V, \cdots, V] \subseteq I$. If I satisfies $[I, I, V, \cdots, V] = 0$, then I is called an *abelian ideal.*

The center of an n-Lie algebra V is

$$Z(V) = \{x \in V \mid [x, V, \cdots, V] = 0\}.$$

It is clear that $Z(V)$ is an abelian ideal of V.

A metric n-Lie algebra [11] is an n-Lie algebra V that has a nondegenerate symmetric bilinear form ρ on V, which is invariant,

$$\rho([x_1, \cdots, x_{n-1}, y_1], y_2) + \rho([x_1, \cdots, x_{n-1}, y_2], y_1) = 0, \quad \text{for all } x_i, y_j \in V.$$

Such a bilinear form ρ is called an *invariant scalar product* on V or a *metric* on V. Note the ρ is not necessarily positive definite.

Let W be a subspace of a metric n-Lie algebra V. *The orthogonal complement* of W is defined by

$$W^{\perp} = \{x \in V \mid \rho(w, x) = 0 \text{ for all } w \in W\}.$$

If W is an ideal, then $W^{\perp}$ is also an ideal and $(W^{\perp})^{\perp} = W$, and $\dim W + \dim W^{\perp} = \dim V$.

In the following, we suppose that F is a field of characteristic zero.

An N-order cubic matrix $A=(a_{ijk})$ (see [10]) over a field F is an ordered object which the elements with 3 indices, and the element in the position (i,j,k) is $(A)_{ijk}=a_{ijk}$, $a_{ijk}\in F$, $1\le i,j,k\le N$. Denote the set of all cubic matrix over a field F by Ω. Then Ω is an N^3-dimensional vector space with

$$A+B=(a_{ijk}+b_{ijk})\in\Omega,\quad \lambda A=(\lambda a_{ijk})\in\Omega,$$

for $\forall A=(a_{ijk}),B=(b_{ijk})\in\Omega,\ \lambda\in F$, that is, $(A+B)_{ijk}=a_{ijk}+b_{ijk}$, $(\lambda A)_{ijk}=\lambda a_{ijk}$.

Denote E_{ijk} a cubic matrix with the element in the position (i,j,k) is 1 and elsewhere are zero, that is, $E_{ijk}=(a_{lmn})$ with $a_{lmn}=\delta_{il}\delta_{jm}\delta_{kn}$, $1\le l,m,n\le N$, and the cubic matrix $E_i=\sum\limits_{j=1}^{N}E_{ijj}$, $1\le i\le N$. Then $\{E_{ijk},\ 1\le i,j,k\le N\}$ is a basis of Ω, and for every $A=(a_{ijk})\in\Omega$, $A=\sum\limits_{1\le i,j,k\le N}a_{ijk}E_{ijk},\ a_{ijk}\in F$.

Every cubic matrix A can be written as the type of blocking form

$$A=(A_1^1,\cdots,A_N^1),\ \ A_i^1=(a_{ijk}),1\le i\le N, \tag{1.1}$$

where $A_i^1=(a_{ijk})$, $1\le i\le N$ are usual $(N\times N)$-order matrices with the element a_{ijk} at the position of the j^{th}-row and the k^{th}-column.

The paper [10] defined the multiplication $*_{11}$ on Ω and the trace function $\langle A\rangle_1$ of cubic matrices

$$(A*_{11}B)_{ijk}=\sum_{p=1}^{N}a_{ijp}b_{ipk},\quad \langle A\rangle_1=\sum_{i,p=1}^{N}a_{ipp}. \tag{1.2}$$

Then for $A=(A_1^1,\cdots,A_N^1)$, $B=(B_1^1,\cdots,B_N^1)\in\Omega$,

$$A*_{11}B=(A_1^1B_1^1,\cdots,A_N^1B_N^1). \tag{1.3}$$

$(\Omega,*_{11})$ is an associative algebra and satisfies $\langle A*_{11}B\rangle_1=\langle B*_{11}A\rangle_1$ for $A,B\in\Omega$. By [8], Ω is a 3-Lie algebra in the multiplication for $A,B,C\in\Omega$,

$$[A,B,C]_1=\langle A\rangle_1(B*_{11}C-C*_{11}B)$$
$$+\langle B\rangle_1(C*_{11}C-A*_{11}B+\langle C\rangle_1(A*_{11}C-B*_{11}B).$$

Lemma 1.1[10] The 3-Lie algebra $(\Omega,[,,]_1)$ is an indecomposable 3-Lie algebra, and has a semi-direct product $\Omega=I\dot{+}FE_1$, where I is a maximal ideal of $(\Omega,[,,]_1)$ with the codimension one and $E_1=\sum\limits_{j=1}^{N}E_{1jj}$.

2 3-Lie Algebra $(\Omega,[,,]_{tr_1})$

For promoting the 3-Lie structure on Ω, we define the new trace function tr_1 of cubic matrix. For $A=(A_1^1,\cdots,A_N^1)\in\Omega$, defines

$$tr_1(A)=tr(A_1)=\sum_{p=1}^{N}a_{1pp}. \tag{2.1}$$

Then we have the following result.

Theorem 2.1. $(\Omega, [, ,]_{tr_1})$ is a 3-Lie algebra in the multiplication

$$[A, B, C]_{tr_1} = tr_1(A)(B *_{11} C - C *_{11} B) \\ +tr_1(B)(C *_{11} C - A *_{11} B + tr_1(C)(A *_{11} C - B *_{11} B). \quad (2.2)$$

And the multiplication in the basis $\{E_{ijk},\ 1 \leq i, j, k \leq N\}$ is as follows

$$[E_{ijk}, E_{lmn}, E_{pqr}] = \delta_{i1}\delta_{jk}\delta_{lp}(\delta_{nq}E_{lmr} - \delta_{rm}E_{lqn}) + \delta_{l1}\delta_{mn}\delta_{pi}(\delta_{rj}E_{pqk} - \delta_{kq}E_{pjr}) \\ +\delta_{p1}\delta_{qr}\delta_{il}(\delta_{km}E_{ijn} - \delta_{jn}E_{imk}). \quad (2.3)$$

Proof By Eqs. (1.3) and (2.1), $\forall A = (A_1^1, \cdots, A_N^1), B = (B_1^1, \cdots, B_N^1) \in \Omega$, $tr_1(A *_{11} B) = tr(A_1^1 B_1^1) = tr(B_1^1 A_1^1) = tr_1(B *_{11} A)$. Then by paper [8], $(\Omega, [, ,]_{tr_1})$ is a 3-Lie algebra in the multiplication (2.2). By Eqs. (1.2), (2.1) and (2,2), we obtain Eq. (2.3).

Now we study the structure of the 3-Lie algebra $(\Omega, [, ,]_{tr_1})$.

Theorem 2.2 The 3-Lie algebra $(\Omega, [, ,]_{tr_1})$ can be decomposed into the direct sum of subalgebras

$$\Omega = \Omega_1 \dot{+} \cdots \dot{+} \Omega_N,$$

where $\Omega_i = \{A = (A_1^1, \cdots, A_i^1, \cdots, A_N^1) \in \Omega \mid A = (0, \cdots, 0, A_i^1, 0, \cdots, 0)\}$, $1 \leq i \leq N$. Ω_j, $2 \leq j \leq N$ are abelian subalgebras, but Ω_1 is a non-abelian subalgebra. And $\Omega_1 \dot{+} \Omega_{j_1} \dot{+} \cdots \dot{+} \Omega_{j_t}$, $2 \leq j_1 < \cdots < j_t \leq N$, are non-abelian subalgebras.

Proof For every $2 \leq j \leq N$, and $A = (0, \cdots, 0, A_j^1, 0, \cdots, 0)$, $B = (0, \cdots, 0, B_j^1, 0, \cdots, 0)$, $C = (0, \cdots, 0, C_j^1, 0, \cdots, 0) \in \Omega_j$, by Theorem 2.1 and Eq (2.1), $tr_1(A) = tr_1(B) = tr_1(C) = 0$, therefore, $[A, B, C]_{tr_1} = 0$. Then Ω_j, $2 \leq j \leq N$ are abelian subalgebras.

Since there exists $A = (A_1^1, 0, \cdots, 0) \in \Omega_1$, such that $tr_1(A) = tr(A_1^1) \neq 0$, by Eqs.(2.2), Ω_1 is a non-abelian subalgebra. Similarly, for the cases $\Omega_1 \dot{+} \Omega_{j_1} \dot{+} \cdots \dot{+} \Omega_{j_t}$, $2 \leq j_1 < \cdots < j_t \leq N$. The result follows.

Theorem 2.3 Let $(\Omega, [, ,]_{tr_1})$ be the 3-Lie algebra in Theorem 2.1. Then we have

1). The derived algebra $\Omega^1 = [\Omega, \Omega, \Omega]_{tr_1}$ has dimension $N^3 - N$, and

$$\Omega^1 = \{A = (A_1^1, A_2^1, \cdots, A_N^1) \in \Omega \mid tr(A_i^1) = 0, 1 \leq i \leq N\}.$$

2). $\Omega_0 = \{A = (0, A_2^1, \cdots, A_N^1) \in \Omega\}$ is a non-abelian ideal of the 3-Lie algebra, but Ω_0 is a maximal abelian subalgebra.

3). $\overline{\Omega} = \{A = (A_1^1, A_2^1, \cdots, A_N^1) \in \Omega \mid tr(A_1^1) = 0\}$ is the maximal ideal of the 3-Lie algebra with codimension one.

4). The center $Z(\Omega)$ of the 3-Lie algebra $(\Omega, [, ,]_{tr_1})$ has dimension $N - 1$, and

$$Z(\Omega) = \{A = (0, a_2E, \cdots, a_NE) \in \Omega \mid a_2, \cdots, a_N \in F\},$$

where E is the $(N \times N)$-order unit matrix.

5) $(\Omega, [,,]_{tr_1})$ is decomposable, and has a decomposition

$$\Omega = I \oplus Z(\Omega),$$

where $I = \Omega^1 + FE_1$ is an $(N^3 - N + 1)$-dimensional ideal of Ω, $I \cap Z(\Omega) = 0$, where $E_1 = (E, 0, \cdots, 0)$.

Proof By Eqs. (1.3) and (2.2), for $A = (A_1^1, \cdots, A_N^1)$, $B = (B_1^1, \cdots, B_N^1)$, $C = (C_1^1, \cdots, C_N^1) \in \Omega$,

$$\begin{aligned}[A,B,C]_{tr_1} &= tr_1(A)(\ B_1^1C_1^1 - C_1^1B_1^1,\ \cdots,\ B_N^1C_N^1 - C_N^1B_N^1\ )\\ &+tr_1(B)(\ C_1^1A_1^1 - A_1^1C_1^1,\ \cdots,\ C_N^1A_N^1 - A_N^1C_N^1\ )\\ &+tr_1(C)(\ A_1^1B_1^1 - B_1^1A_1^1,\ \cdots,\ A_N^1B_N^1 - B_N^1A_N^1\ ).\end{aligned}$$

Since for $1 \le i \le N$,

$$tr(B_i^1C_i^1 - C_i^1B_i^1) = tr(A_i^1B_i^1 - B_i^1A_i^1) = tr(C_i^1A_i^1 - A_i^1C_i^1) = 0,$$

the result 1) follows. The result 2) follows from the direct computation.

It is clear, $\dim \overline{\Omega} = N^3 - 1$. Follows from the result 1), $\overline{\Omega}$ contains the derived algebra Ω^1. Therefore, $\overline{\Omega}$ is a maximal ideal of the 3-Lie algebra. The result 3) follows.

Let $A = (A_1^1, \cdots, A_N^1) \in \Omega$ be in the center of the 3-Lie algebra $(\Omega, [,,]_{tr_1})$. Since for every $i \ge 2$,

$[(E_{11}, 0, \cdots, 0), (0, \cdots, 0, E_{ii}, 0, \cdots, 0), A]_{tr_1}$
$= tr_1((E_{11}, 0, \cdots, 0))[(0, \cdots, 0, E_{ii}, 0, \cdots, 0) *_{11} A - A *_{11} (0, \cdots, 0, E_{ii}, 0, \cdots, 0)]$
$+tr_1(A)[(E_{11}, 0, \cdots, 0) *_{11} (0, \cdots, 0, E_{ii}, 0, \cdots, 0) - (0, \cdots, 0, E_{ii}, 0, \cdots, 0) *_{11} (E_{11}, 0, \cdots, 0)] = (0, \cdots, 0, E_{ii}A_i^1 - A_i^1E_{ii}, 0, \cdots, 0) = 0,$
we obtain $A_i^1 = a_iE$, where E_{ii}, $1 \le i \le N$ are $(N \times N)$-order matrices with 1 at the position i^{th}-row and i^{th}-column and others are zero, $E = \sum\limits_{i=1}^{N} E_{ii}$ is the $(N \times N)$-order unit matrix. Then, the cubic matrix A has the form $A = (A_1^1, a_2E, \cdots, a_NE)$. For $1 \le i \ne j, j \ne k, i \ne k \le N$, since

$[A, (E_{ij}, 0 \cdots, 0), (E_{jk}, 0, \cdots, 0)]$
$= [(A_1^1, E, \cdots, E), (E_{ij}, 0 \cdots, 0), (E_{jk}, 0, \cdots, 0)]$
$= tr(A_1^1)(E_{ik}, 0, \cdots, 0) = = (0, \cdots, 0),$
we have $tr(A_1^1) = 0$. For every $1 \le i \ne j \le N$,

$[A, (E_{ij}, 0 \cdots, 0), (E_{11}, 0 \cdots, 0)]$
$= (A_1^1E_{ij} - E_{ij}A_1^1, 0, \cdots, 0) = ((a_{ii} - a_{jj})E_{ij}, 0, \cdots, 0) = 0,$
and $A_1^1 = aE$. Thanks to $tr(A_1^1) = Na = 0$, $A_1^1 = 0$. The result 4) follows.

Clearly, $I = \Omega^1 + FE_1$ is an ideal of the 3-Lie algebra $(\Omega, [,,]_{tr_1})$ since $I \supset \Omega^1$. By result 1) and result 4), $I \cap Z(\Omega) = 0$, $\dim I + \dim Z(\Omega) = N^3$, and $[I, Z(\Omega), \Omega]_{tr_1} = 0$. Therefore, $\Omega = I \oplus Z(\Omega)$. The result 5) follows.

Theorem 2.4 There does not exist metric structure on the 3-Lie algebra $(\Omega, [,,]_{tr_1})$.

Proof If ρ is a metric on the 3-Lie algebra $(\Omega, [, ,]_{tr_1})$. Then ρ is a nondegenerate symmetric bilinear form on Ω satisfying

$$\rho([A,B,C]_{tr_1}, D) + \rho(C, [A,B,D]_{tr_1}) = 0, \forall A,B,C,D \in \Omega.$$

Thanks to Lemma 2.3 in paper [11], $Z(\Omega)^{\perp} = \Omega^1$. Then we have

$$\dim \Omega^1 + \dim Z(\Omega) = N^3 - N + (N-1) = N^3 - 1 = N^3.$$

Contradiction. It follows the result.

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