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The k-Analogues of Some Inequalities for the Digamma Function

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Abstract

In this paper, we present the k-analogues of some inequalities concerning the digamma function.

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1 Introduction and Preliminaries

We begin by recalling some basic definitions involving the Gamma function.

The classical Euler's Gamma function, $\Gamma(t)$ is commonly defined by

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx, \qquad t > 0.$$

The digamma function, $\psi(t)$ is defined as the logarithmic derivative of the Gamma function, that is,

$$\psi(t) = \frac{d}{dt} \ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0.$$

The k-analogue of the Gamma function $\Gamma_k(t)$ is also defined by (see [1],[2])

$$\Gamma_k(t) = \int_0^\infty e^{-\frac{x^k}{k}} x^{t-1}\, dx, \quad k > 0, \quad t > 0.$$

Equivalently, the k-digamma function, $\psi_k(t)$ is defined as

$$\psi_k(t) = \frac{d}{dt}\ln(\Gamma_k(t)) = \frac{\Gamma_k'(t)}{\Gamma_k(t)}, \quad t > 0.$$

The functions $\psi(t)$ and $\psi_k(t)$ as defined above have the following series representations.

$$\psi(t) = -\gamma + (t-1)\sum_{n=0}^{\infty}\frac{1}{(1+n)(n+t)}, \quad t > 0$$

$$\psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty}\frac{t}{nk(nk+t)}, \quad k > 0, \quad t > 0$$

where γ is the Euler-Mascheroni's constant. For some properties of these functions, see [4], [2] and the references therein.

By taking the m-th derivative of the functions $\psi(t)$ and $\psi_k(t)$, it can be shown that the following statements are valid for $m \in N$.

$$\psi^{(m)}(t) = (-1)^{m+1}m!\sum_{n=0}^{\infty}\frac{1}{(n+t)^{m+1}}, \quad t > 0$$

$$\psi_k^{(m)}(t) = (-1)^{m+1}m!\sum_{n=0}^{\infty}\frac{1}{(nk+t)^{m+1}}, \quad k > 0, \quad t > 0.$$

In a recent paper [3], Sulaiman presented the following results.

$$\psi(s+t) \geq \psi(s) + \psi(t) \tag{1}$$

where $t > 0$ and $0 < s < 1$.

$$\psi^{(m)}(s+t) \leq \psi^{(m)}(s) + \psi^{(m)}(t) \tag{2}$$

where m is a positive odd integer and $s, t > 0$.

$$\psi^{(m)}(s+t) \geq \psi^{(m)}(s) + \psi^{(m)}(t) \tag{3}$$

where m is a positive even integer and $s, t > 0$.

The objective of this paper is to establish that the inequalities (1), (2) and (3) still hold true for the function $\psi_k(t)$.

2 Main Results

We now present the results of this paper.

Theorem 2.1. *Let $t > 0$, $0 < s \leq 1$ and $k > 0$. Then the following inequality is valid.*

$$\psi_k(s+t) \geq \psi_k(s) + \psi_k(t). \tag{4}$$

Proof. Let $\mu(t) = \psi_k(s+t) - \psi_k(s) - \psi_k(t)$. Then fixing s we have,

$$\begin{aligned}\mu'(t) = \psi_k'(s+t) - \psi_k'(t) &= \sum_{n=0}^{\infty} \frac{1}{(nk+s+t)^2} - \sum_{n=0}^{\infty} \frac{1}{(nk+t)^2} \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{(nk+s+t)^2} - \frac{1}{(nk+t)^2} \right] \leq 0\end{aligned}$$

That implies μ is non-increasing. Also,

$$\begin{aligned}\lim_{t\to\infty} \mu(t) &= \lim_{t\to\infty} [\psi_k(s+t) - \psi_k(s) - \psi_k(t)] \\ &= \lim_{t\to\infty} \left[\frac{1}{s} + \frac{1}{t} - \frac{1}{s+t} - \frac{\ln k - \gamma}{k} \right. \\ &\qquad \left. + \sum_{n=1}^{\infty} \left(\frac{s+t}{nk(nk+s+t)} - \frac{s}{nk(nk+s)} - \frac{t}{nk(nk+t)} \right) \right] \\ &= \frac{1}{s} - \frac{\ln k - \gamma}{k} + \sum_{n=1}^{\infty} \left(\frac{1}{nk} - \frac{s}{nk(nk+s)} - \frac{1}{nk} \right) \\ &= \frac{1}{s} - \frac{\ln k - \gamma}{k} - \sum_{n=1}^{\infty} \frac{s}{nk(nk+s)} \\ &= -\psi_k(s) \geq 0. \quad (\text{Note that } \psi_k(s) < 0 \text{ for } 0 < s \leq 1)\end{aligned}$$

Therefore $\mu(t) \geq 0$ and hence the proof.

Theorem 2.2. *Let $s, t > 0$ and $k > 0$. Suppose that m is a positive odd integer, then the following inequality is valid.*

$$\psi_k^{(m)}(s+t) \leq \psi_k^{(m)}(s) + \psi_k^{(m)}(t). \tag{5}$$

Proof. Let $\eta(t) = \psi_k^{(m)}(s+t) - \psi_k^{(m)}(s) - \psi_k^{(m)}(t)$. Then fixing s we have,

$$\begin{aligned}\eta'(t) &= \psi_k^{(m+1)}(s+t) - \psi_k^{(m+1)}(t) \\ &= (-1)^{m+2}(m+1)! \sum_{n=0}^{\infty} \left[\frac{1}{(nk+s+t)^{m+2}} - \frac{1}{(nk+t)^{m+2}} \right] \\ &= -(m+1)! \sum_{n=0}^{\infty} \left[\frac{1}{(nk+s+t)^{m+2}} - \frac{1}{(nk+t)^{m+2}} \right] \geq 0. \text{ (since } m \text{ is odd)}\end{aligned}$$

That implies η is non-decreasing. Also,

$$\begin{aligned}\lim_{t\to\infty}\eta(t) &= (-1)^{m+1}m!\lim_{t\to\infty}\sum_{n=0}^{\infty}\left[\frac{1}{(nk+s+t)^{m+1}}-\frac{1}{(nk+s)^{m+1}}-\frac{1}{(nk+t)^{m+1}}\right]\\ &= (-1)^{m+1}m!\sum_{n=0}^{\infty}\left[-\frac{1}{(nk+s)^{m+1}}\right]\\ &= m!\sum_{n=0}^{\infty}\left[-\frac{1}{(nk+s)^{m+1}}\right]\leq 0. \quad \text{(since } m \text{ is odd)}\end{aligned}$$

Therefore $\eta(t)\leq 0$ and hence the proof.

Theorem 2.3. *Let $s,t>0$ and $k>0$. Suppose that m is a positive even integer, then the following inequality is valid.*

$$\psi_k^{(m)}(s+t)\geq\psi_k^{(m)}(s)+\psi_k^{(m)}(t). \tag{6}$$

Proof. Let $\lambda(t)=\psi_k^{(m)}(s+t)-\psi_k^{(m)}(s)-\psi_k^{(m)}(t)$. Then by fixing s we have,

$$\begin{aligned}\lambda'(t) &= \psi_k^{(m+1)}(s+t)-\psi_k^{(m+1)}(t)\\ &= (-1)^{m+2}(m+1)!\sum_{n=0}^{\infty}\left[\frac{1}{(nk+s+t)^{m+2}}-\frac{1}{(nk+t)^{m+2}}\right]\\ &= (m+1)!\sum_{n=0}^{\infty}\left[\frac{1}{(nk+s+t)^{m+2}}-\frac{1}{(nk+t)^{m+2}}\right]\leq 0. \text{ (since } m \text{ is even)}\end{aligned}$$

That implies λ is non-increasing. Also,

$$\begin{aligned}\lim_{t\to\infty}\lambda(t) &= (-1)^{m+1}m!\lim_{t\to\infty}\sum_{n=0}^{\infty}\left[\frac{1}{(nk+s+t)^{m+1}}-\frac{1}{(nk+s)^{m+1}}-\frac{1}{(nk+t)^{m+1}}\right]\\ &= (-1)^{m+1}m!\sum_{n=0}^{\infty}\left[-\frac{1}{(nk+s)^{m+1}}\right]\\ &= -m!\sum_{n=0}^{\infty}\left[-\frac{1}{(nk+s)^{m+1}}\right]\geq 0. \quad \text{(since } m \text{ is even)}\end{aligned}$$

Therefore $\lambda(t)\geq 0$ and hence the proof.

References

[1] R. Díaz and E. Pariguan, *On hypergeometric functions and Pachhammer k-symbol*, Divulgaciones Matemtícas **15**(2)(2007), 179-192.

[2] F. Merovci, *Power Product Inequalities for the* Γ_k *Function*, Int. Journal of Math. Analysis **4**(21)(2010), 1007-1012.

[3] W. T. Sulaiman, *Turan inequalites for the digamma and polygamma functions*, South Asian J. Math. **1**(2)(2011), 49-55.

[4] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Camb. Univ. Press, 1969.

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