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The p-Analogues of Some Inequalities for the Digamma Function

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Abstract

In this paper, we present the p-analogues of some inequalities concerning the digamma function.

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1 Introduction and Preliminaries

We begin by recalling some basic definitions involving the Gamma function.

The classical Euler's Gamma function, $\Gamma(t)$ is commonly defined by

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx, \qquad t > 0.$$

The digamma function, $\psi(t)$ is defined as the logarithmic derivative of the Gamma function, that is,

$$\psi(t) = \frac{d}{dt} \ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0.$$

The p-analogue of the Gamma function, $\Gamma_p(t)$ is defined by (see [1],[2])

$$\Gamma_p(t) = \frac{p!p^t}{t(t+1)\dots(t+p)} = \frac{p^t}{t(1+\frac{t}{1})\dots(1+\frac{t}{p})}, \quad p \in N, \quad t > 0.$$

Similarly, the p-digamma function, $\psi_p(t)$ is defined as,

$$\psi_p(t) = \frac{d}{dt}\ln(\Gamma_p(t)) = \frac{\Gamma'_p(t)}{\Gamma_p(t)}, \qquad t > 0.$$

The functions $\psi(t)$ and $\psi_p(t)$ as defined above have the following series representations.

$$\psi(t) = -\gamma + (t-1)\sum_{n=0}^{\infty}\frac{1}{(1+n)(n+t)}, \quad t > 0.$$

$$\psi_p(t) = \ln p - \sum_{n=0}^{p}\frac{1}{n+t}, \quad p \in N, \quad t > 0.$$

where γ is the Euler-Mascheroni's constant. For some properties of these functions, see [4], [2] and the references therein.

By taking the m-th derivative of the functions $\psi(t)$ and $\psi_p(t)$, it can be shown that the following statements are valid for $m \in N$.

$$\psi^{(m)}(t) = (-1)^{m+1}m!\sum_{n=0}^{\infty}\frac{1}{(n+t)^{m+1}}, \quad t > 0.$$

$$\psi_p^{(m)}(t) = (-1)^{m-1}m!\sum_{n=0}^{p}\frac{1}{(n+t)^{m+1}}, \quad p \in N, \quad t > 0.$$

In a recent paper [3], Sulaiman presented the following results.

$$\psi(s+t) \geq \psi(s) + \psi(t) \tag{1}$$

where $t > 0$ and $0 < s < 1$.

$$\psi^{(m)}(s+t) \leq \psi^{(m)}(s) + \psi^{(m)}(t) \tag{2}$$

where m is a positive odd integer and $s, t > 0$.

$$\psi^{(m)}(s+t) \geq \psi^{(m)}(s) + \psi^{(m)}(t) \tag{3}$$

where m is a positive even integer and $s, t > 0$.

The objective of this paper is to establish that the inequalities (1), (2) and (3) still hold true for the function $\psi_p(t)$.

2 Main Results

We now present the results of this paper.

Theorem 2.1. *Let $t > 0$, $0 < s \leq 1$ and $p \in N$. Then the following inequality holds true.*

$$\psi_p(s+t) \geq \psi_p(s) + \psi_p(t). \tag{4}$$

Proof. Let $f(t) = \psi_p(s+t) - \psi_p(s) - \psi_p(t)$. Then fixing s we have,

$$\begin{aligned} f'(t) = \psi_p'(s+t) - \psi_p'(t) &= \sum_{n=0}^{p} \frac{1}{(n+s+t)^2} - \sum_{n=0}^{p} \frac{1}{(n+t)^2} \\ &= \sum_{n=0}^{p} \left[\frac{1}{(n+s+t)^2} - \frac{1}{(n+t)^2} \right] \leq 0 \end{aligned}$$

That implies f is non-increasing. Further,

$$\begin{aligned} \lim_{t\to\infty} f(t) &= \lim_{t\to\infty} \left[\psi_p(s+t) - \psi_p(s) - \psi_p(t)\right] \\ &= \lim_{t\to\infty} \left[-\ln p + \sum_{n=0}^{p} \frac{1}{(n+s)} + \sum_{n=0}^{p} \frac{1}{(n+t)} - \sum_{n=0}^{p} \frac{1}{(n+s+t)} \right] \\ &= -\ln p + \lim_{t\to\infty} \sum_{n=0}^{p} \left[\frac{1}{(n+s)} + \frac{1}{(n+t)} - \frac{1}{(n+s+t)} \right] \\ &= -\ln p + \sum_{n=0}^{p} \frac{1}{(n+s)} \geq 0. \end{aligned}$$

Therefore $f(t) \geq 0$ yielding the result.

Theorem 2.2. *Let $s, t > 0$ and $p \in N$. Suppose that m is a positive odd integer, then the following inequality holds true.*

$$\psi_p^{(m)}(s+t) \leq \psi_p^{(m)}(s) + \psi_p^{(m)}(t). \tag{5}$$

Proof. Let $g(t) = \psi_p^{(m)}(s+t) - \psi_p^{(m)}(s) - \psi_p^{(m)}(t)$. Then fixing s we have,

$$\begin{aligned} g'(t) &= \psi_p^{(m+1)}(s+t) - \psi_p^{(m+1)}(t) \\ &= (-1)^m (m+1)! \sum_{n=0}^{p} \left[\frac{1}{(n+s+t)^{m+2}} - \frac{1}{(n+t)^{m+2}} \right] \\ &= -(m+1)! \sum_{n=0}^{p} \left[\frac{1}{(n+s+t)^{m+2}} - \frac{1}{(n+t)^{m+2}} \right], \text{ (since } m \text{ is odd)} \\ &= (m+1)! \sum_{n=0}^{p} \left[\frac{1}{(n+t)^{m+2}} - \frac{1}{(n+s+t)^{m+2}} \right] \geq 0. \end{aligned}$$

That implies g is non-decreasing. Further,

$$\begin{aligned}\lim_{t\to\infty} g(t) &= \lim_{t\to\infty}\left[\psi_p^{(m)}(s+t)-\psi_p^{(m)}(s)-\psi_p^{(m)}(t)\right]\\
&= (-1)^{m-1}m!\lim_{t\to\infty}\sum_{n=0}^{p}\left[\frac{1}{(n+s+t)^{m+1}}-\frac{1}{(n+s)^{m+1}}-\frac{1}{(n+t)^{m+1}}\right]\\
&= m!\lim_{t\to\infty}\sum_{n=0}^{p}\left[\frac{1}{(n+s+t)^{m+1}}-\frac{1}{(n+s)^{m+1}}-\frac{1}{(n+t)^{m+1}}\right], \text{ (for odd } m)\\
&= -m!\sum_{n=0}^{p}\frac{1}{(n+s)^{m+1}}\leq 0.\end{aligned}$$

Therefore $g(t)\leq 0$ yielding the result.

Theorem 2.3. *Let $s,t>0$ and $p\in N$. Suppose that m is a positive even integer, then the following inequality holds true.*

$$\psi_p^{(m)}(s+t)\geq \psi_p^{(m)}(s)+\psi_p^{(m)}(t). \tag{6}$$

Proof. Let $h(t)=\psi_p^{(m)}(s+t)-\psi_p^{(m)}(s)-\psi_p^{(m)}(t)$. Then by fixing s we have,

$$\begin{aligned}h'(t) &= \psi_p^{(m+1)}(s+t)-\psi_p^{(m+1)}(t)\\
&= (-1)^{m}(m+1)!\sum_{n=0}^{p}\left[\frac{1}{(n+s+t)^{m+2}}-\frac{1}{(n+t)^{m+2}}\right]\\
&= (m+1)!\sum_{n=0}^{p}\left[\frac{1}{(n+s+t)^{m+2}}-\frac{1}{(n+t)^{m+2}}\right], \text{ (since } m \text{ is even)}\\
&= (m+1)!\sum_{n=0}^{p}\left[\frac{1}{(n+s+t)^{m+2}}-\frac{1}{(n+t)^{m+2}}\right]\leq 0.\end{aligned}$$

That implies h is non-increasing. Further,

$$\begin{aligned}\lim_{t\to\infty} h(t) &= \lim_{t\to\infty}\left[\psi_p^{(m)}(s+t)-\psi_p^{(m)}(s)-\psi_p^{(m)}(t)\right]\\
&= (-1)^{m-1}m!\lim_{t\to\infty}\sum_{n=0}^{p}\left[\frac{1}{(n+s+t)^{m+1}}-\frac{1}{(n+s)^{m+1}}-\frac{1}{(n+t)^{m+1}}\right]\\
&= -m!\lim_{t\to\infty}\sum_{n=0}^{p}\left[\frac{1}{(n+s+t)^{m+1}}-\frac{1}{(n+s)^{m+1}}-\frac{1}{(n+t)^{m+1}}\right], \text{ (for even } m)\\
&= m!\sum_{n=0}^{p}\frac{1}{(n+s)^{m+1}}\geq 0.\end{aligned}$$

Therefore $h(t)\geq 0$ yielding the result.

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