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The q-Analogues of Some Inequalities for the Digamma Function

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Abstract

In this paper, we present the q-analogues of some inequalities concerning the digamma function.

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1 Introduction and Preliminaries

We begin by recalling some basic definitions involving the Gamma function.

The classical Euler's Gamma function, $\Gamma(t)$ is commonly defined by

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx, \qquad t > 0.$$

The digamma function, $\psi(t)$ is defined as the logarithmic derivative of the Gamma functio, that is,

$$\psi(t) = \frac{d}{dt} \ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0.$$

The q-analogue of the Gamma function, $\Gamma_q(t)$ is defined by (see [2])

$$\Gamma_q(t) = (1-q)^{1-t} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{t+n}}, \quad q \in (0,1), \quad t > 0.$$

Similarly, the q-digamma function, $\psi_q(t)$ is defined as,

$$\psi_q(t) = \frac{d}{dt} \ln(\Gamma_q(t)) = \frac{\Gamma_q'(t)}{\Gamma_q(t)}, \quad t > 0.$$

The functions $\psi(t)$ and $\psi_q(t)$ as defined above have the following series representations.

$$\psi(t) = -\gamma + (t-1) \sum_{n=0}^{\infty} \frac{1}{(1+n)(n+t)}, \quad t > 0.$$

$$\psi_q(t) = -\ln(1-q) + (\ln q) \sum_{n=1}^{\infty} \frac{q^{nt}}{1-q^n}, \quad q \in (0,1), \quad t > 0.$$

where γ is the Euler-Mascheroni's constant. For some properties of these functions, see [4], [1] and the references therein.

By taking the m-th derivative of the functions $\psi(t)$ and $\psi_q(t)$, it can be shown that the following statements are valid for $m \in N$.

$$\psi^{(m)}(t) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(n+t)^{m+1}}, \quad t > 0.$$

$$\psi_q^{(m)}(t) = (\ln q)^{m+1} \sum_{n=1}^{\infty} \frac{n^m q^{nt}}{1-q^n}, \quad q \in (0,1), \quad t > 0.$$

In a recent paper [3], Sulaiman presented the following results.

$$\psi(s+t) \geq \psi(s) + \psi(t) \tag{1}$$

where $t > 0$ and $0 < s < 1$.

$$\psi^{(m)}(s+t) \leq \psi^{(m)}(s) + \psi^{(m)}(t) \tag{2}$$

where m is a positive odd integer and $s, t > 0$.

$$\psi^{(m)}(s+t) \geq \psi^{(m)}(s) + \psi^{(m)}(t) \tag{3}$$

where m is a positive even integer and $s, t > 0$.

The objective of this paper is to establish that the inequalities (1), (2) and (3) still hold true for the function $\psi_q(t)$.

2 Main Results

We now present the results of this paper.

Theorem 2.1. *Let $t > 0$, $0 < s \leq 1$ and $q \in (0,1)$. Then the following inequality is valid.*

$$\psi_q(s+t) \geq \psi_q(s) + \psi_q(t). \tag{4}$$

Proof. Let $u(t) = \psi_q(s+t) - \psi_q(s) - \psi_q(t)$. Then fixing s we have,

$$\begin{aligned}
u'(t) = \psi_q'(s+t) - \psi_q'(t) &= (\ln q)^2 \sum_{n=1}^{\infty} \left[\frac{nq^{n(s+t)}}{1-q^n} - \frac{nq^{nt}}{1-q^n} \right] \\
&= (\ln q)^2 \sum_{n=1}^{\infty} \left[\frac{nq^{n(s+t)} - nq^{nt}}{1-q^n} \right] \\
&= (\ln q)^2 \sum_{n=1}^{\infty} \frac{nq^{nt}(q^{ns}-1)}{1-q^n} \leq 0.
\end{aligned}$$

That implies u is non-increasing. In addition,

$$\begin{aligned}
\lim_{t\to\infty} u(t) &= \lim_{t\to\infty} [\psi_q(s+t) - \psi_q(s) - \psi_q(t)] \\
&= \ln(1-q) + (\ln q) \lim_{t\to\infty} \sum_{n=1}^{\infty} \left[\frac{q^{n(s+t)}}{1-q^n} - \frac{q^{ns}}{1-q^n} - \frac{q^{nt}}{1-q^n} \right] \\
&= \ln(1-q) + (\ln q) \lim_{t\to\infty} \sum_{n=1}^{\infty} \left[\frac{q^{n(s+t)} - q^{ns} - q^{nt}}{1-q^n} \right] \\
&= \ln(1-q) + (\ln q) \lim_{t\to\infty} \sum_{n=1}^{\infty} \left[\frac{q^{ns}.q^{nt} - q^{ns} - q^{nt}}{1-q^n} \right] \\
&= \ln(1-q) - (\ln q) \sum_{n=1}^{\infty} \frac{q^{ns}}{1-q^n} \geq 0.
\end{aligned}$$

Therefore $u(t) \geq 0$ concluding the proof.

Theorem 2.2. *Let $s, t > 0$ and $q \in (0,1)$. Suppose that m is a positive odd integer, then the following inequality is valid.*

$$\psi_q^{(m)}(s+t) \leq \psi_q^{(m)}(s) + \psi_q^{(m)}(t). \tag{5}$$

Proof. Let $v(t) = \psi_q^{(m)}(s+t) - \psi_q^{(m)}(s) - \psi_q^{(m)}(t)$. Then fixing s we have,

$$\begin{aligned}
v'(t) &= \psi_q^{(m+1)}(s+t) - \psi_q^{(m+1)}(t) \\
&= (\ln q)^{m+2} \sum_{n=1}^{\infty} \left[\frac{n^{m+1}q^{n(s+t)}}{1-q^n} - \frac{n^{m+1}q^{nt}}{1-q^n} \right] \\
&= (\ln q)^{m+2} \sum_{n=1}^{\infty} \left[\frac{n^{m+1}q^{nt}(q^{ns}-1)}{1-q^n} \right] \geq 0. \text{ (since } m \text{ is odd)}
\end{aligned}$$

That implies v is non-decreasing. In addition,

$$\begin{aligned}\lim_{t\to\infty} v(t) &= (\ln q)^{m+1} \lim_{t\to\infty} \sum_{n=1}^{\infty} \left[\frac{n^m q^{n(s+t)}}{1-q^n} - \frac{n^m q^{ns}}{1-q^n} - \frac{n^m q^{nt}}{1-q^n} \right] \\ &= (\ln q)^{m+1} \lim_{t\to\infty} \sum_{n=1}^{\infty} \left[\frac{n^m q^{ns}.q^{nt}}{1-q^n} - \frac{n^m q^{ns}}{1-q^n} - \frac{n^m q^{nt}}{1-q^n} \right] \\ &= -(\ln q)^{m+1} \sum_{n=1}^{\infty} \frac{n^m q^{ns}}{1-q^n} \leq 0. \text{ (since } m \text{ is odd)}\end{aligned}$$

Therefore $v(t) \leq 0$ concluding the proof.

Theorem 2.3. *Let $s, t > 0$ and $q \in (0,1)$. Suppose that m is a positive even integer, then the following inequality is valid.*

$$\psi_q^{(m)}(s+t) \geq \psi_q^{(m)}(s) + \psi_q^{(m)}(t). \tag{6}$$

Proof. Let $w(t) = \psi_q^{(m)}(s+t) - \psi_q^{(m)}(s) - \psi_q^{(m)}(t)$. Then by fixing s we have,

$$\begin{aligned} w'(t) &= \psi_q^{(m+1)}(s+t) - \psi_q^{(m+1)}(t) \\ &= (\ln q)^{m+2} \sum_{n=1}^{\infty} \left[\frac{n^{m+1} q^{n(s+t)}}{1-q^n} - \frac{n^{m+1} q^{nt}}{1-q^n} \right] \\ &= (\ln q)^{m+2} \sum_{n=1}^{\infty} \left[\frac{n^{m+1} q^{nt}(q^{ns}-1)}{1-q^n} \right] \leq 0. \text{ (since } m \text{ is even)}\end{aligned}$$

That implies w is non-increasing. In addition,

$$\begin{aligned}\lim_{t\to\infty} w(t) &= (\ln q)^{m+1} \lim_{t\to\infty} \sum_{n=1}^{\infty} \left[\frac{n^m q^{n(s+t)}}{1-q^n} - \frac{n^m q^{ns}}{1-q^n} - \frac{n^m q^{nt}}{1-q^n} \right] \\ &= (\ln q)^{m+1} \lim_{t\to\infty} \sum_{n=1}^{\infty} \left[\frac{n^m q^{ns}.q^{nt}}{1-q^n} - \frac{n^m q^{ns}}{1-q^n} - \frac{n^m q^{nt}}{1-q^n} \right] \\ &= -(\ln q)^{m+1} \sum_{n=1}^{\infty} \frac{n^m q^{ns}}{1-q^n} \geq 0. \text{ (since } m \text{ is even)}\end{aligned}$$

Therefore $w(t) \geq 0$ concluding the proof.

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